Elementary Mechanics

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PREFACE

This text has been prepared for a short course in mechanics. The attempt has been made to present the fundamental principles of mechanics in a simple manner and to give a rigorous discussion of each topic to the extent that it is treated.

The text has been written by the following members of the Department of Mathematics of the U. S. Naval Academy: Messrs. W. J. King, J. B. Eppes, J. A. Bullard, John Tyler, Arthur Kiernan, J. N. Galloway, Alex. Dillingham, G. R. Clements, H. M. Robert, Jr., L. S. Dederick, L. T. Wilson, W. F. Shenton, and R. P. Johnson. Other members of the department have offered suggestions and contributed problems.

The work of editing has been done by Commander H. L. Rice, (Math.), U. S. N., and Messrs. Bullard and Dederick.

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ELEMENTARY MECHANICS

CHAPTER I

VECTORS

1. Vector and Scalar Quantities.—The quantities with which we shall be concerned in the study of Mechanics may be divided into two general classes, namely, those which involve both magnitude and direction and those which involve magnitude only. The former are called vector quantities and the latter scalar quantities.

Scalar quantities are of a type with which we are all familiar, though we may not have designated them by this term; they are quantities which can be measured by means of a scale, hence the name. Temperature, time, density, voltage, mass, and many other quantities of every-day experience are illustrations of quantities which can be measured on one type of scale or another.

But not all quantities can be so measured. Thus, as will be shown as each is treated, force, velocity and acceleration require not only magnitude but direction for their determination. These are examples of vector quantities.

2. Graphical Representation of Vector Quantities.—A vector quantity is represented by the segment of a straight line with an arrow at one end to indicate the sense. In Fig. 1 the vector α is represented by the line AB from the initial point A to the terminus B. The arrow-head indicates that the direction along the line is from A to B. The length AB, or a, represents the magnitude of the vector. The student must clearly bear in mind that AB is not the same vector as BA.

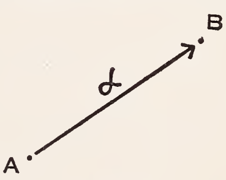

FIG. 1.

3. Equal Vectors.—*Two vectors are said to be equal when they have the same magnitude and direction.* From this it follows readily that we can move a vector at pleasure so long as we do not alter its magnitude or change its direction. In Fig. 2 the vectors α and β are drawn equal in length and parallel in direction; hence we may write $\alpha=\beta$. On the other hand, we say $\gamma=-\alpha$ because the direction of γ is opposite to that of α, although their lengths, and hence their magnitudes, are the same.

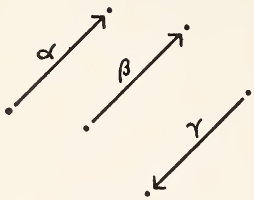

FIG. 2.

4. Multiplication of a Vector Quantity by a Scalar Quantity.—The product of a vector quantity and a scalar quantity is a vector quantity. The resultant vector is one whose direction is the same as that of the given vector and whose magnitude is the magnitude of the given vector multiplied by the scalar.

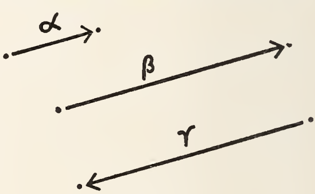

FIG. 3.

If the scalar is negative the sense of the vector direction is reversed.

Thus, if α is the vector which represents the given quantity, $\frac{5}{2}\alpha$ would be represented by β while $-\frac{5}{2}\alpha$ would be represented by γ.

5. Addition of Two Vectors.—Move one of the vectors until its origin falls upon the terminus of the other vector; complete the triangle by drawing a vector, the origin of which coincides with that of the first vector. *This vector is defined as the sum of the two given vectors and is called the resultant.* In Fig. 4, given any two vectors as α and β, move β to the terminus B of α

and draw AC with its sense from A to C. We have then the vector equation

$$\gamma = \alpha + \beta.$$

We could secure this same resultant by what is called the parallelogram method, as follows:

Move one of the vectors until its origin coincides with that of the other vector, complete the parallelogram of which these two vectors form adjacent sides, and then draw a vector which has the common origin of the given vectors for its origin and which is a diagonal of the parallelogram. As an illustration of the parallelogram method consider Fig. 5, in which the two vectors α and β of Fig. 4 are again used. The origin of vector β is put at the origin of vector α and the parallelogram

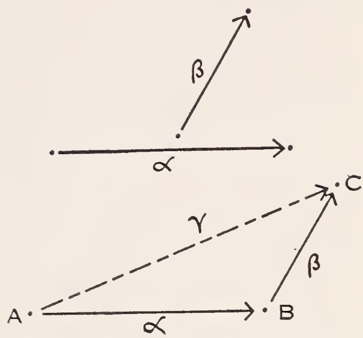

FIG. 4.

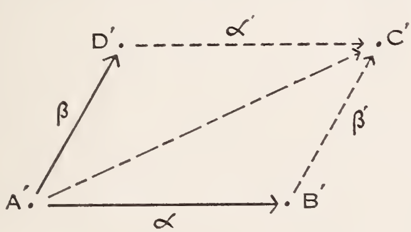

FIG. 5.

$A'B'C'D'$ is completed. Draw the vector $A'C'$. This is the resultant of the two vectors α and β by the parallelogram method. The vector β' is equal to the vector β. Hence the vector triangle

$A'B'C'$ is the same as the vector triangle of Fig. 4 and $A'C'$ is equal to γ.

Further, since α' is at the terminus of β we have the vector equation

$$\beta+\alpha'=\gamma; \text{ similarly } \alpha+\beta'=\gamma$$

But, $\alpha=\alpha'$ and $\beta=\beta'$ and it follows at once that whether α is applied at the terminus of β or *vice versa* the result is the same. Consequently the addition of the vectors is independent of the order in which they are taken.

6. Subtraction of Vectors.—Subtraction in algebra is equivalent to the addition of a negative quantity; likewise *to subtract vectors we reverse the sense of the one to be subtracted and add it to the other.* In Fig. 6 the difference $\alpha-\beta$ of the two vectors is represented by δ. The sum $\alpha+\beta$ is represented by γ. If, however, we draw the other diagonal of the parallelogram, $ABCD$, we can readily see from geometry that DB is equal and parallel to AE. From this it follows at once that the sum and difference of two vectors form the diagonals of the parallelogram determined by them.

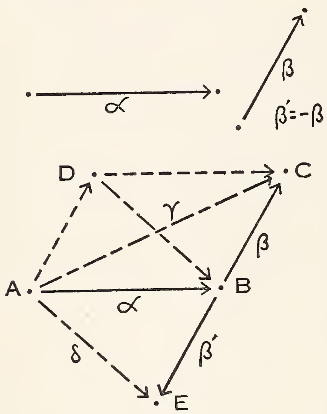

FIG. 6

7. The Addition of Several Vectors consists merely of the successive application of the triangle law for two vectors. Given any system of vectors; in order to find their sum we choose any two of the vectors and secure their resultant. From the above this resultant is unique and is a vector; consequently, when this

resultant is added to any other vector of the system the new resultant is unique. This process is carried on until every vector of the system has been included in the addition.

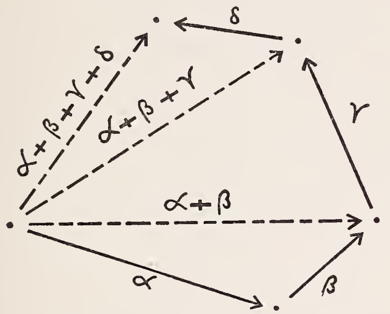

FIG. 7.

Illustration.—To find the sum of the four vectors α, β, γ, δ: We here apply β to the terminus of α and by the triangle method we secure the resultant vector $\alpha+\beta$; to the terminus of this we apply γ and secure $\alpha+\beta+\gamma$; to the terminus of the latter we apply δ and get the resultant of the four vectors, $\alpha+\beta+\gamma+\delta$.

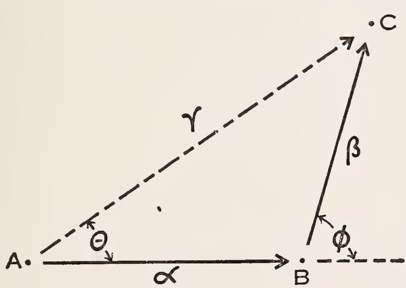

FIG. 8.

The vectors and the resultant together form a polygon. As a matter of practise it is not necessary to draw the diagonals of

the polygon; merely apply β to the terminus of α, γ to that of β, and so forth; complete the polygon by drawing the vector from the starting point to the terminus of the last vector applied.

8. Relations Between the Magnitudes of Two Vectors and Their Resultant.—Let α and β, Fig. 8, be two vectors and γ their resultant. Let θ be the angle between the directions of α and γ and ϕ be the angle between the direction of α and that of β. We will denote the lengths of α, β, and γ by a, b, and c, respectively. Then from trigonometry we have

$$c^2=a^2+b^2+2ab\cdot\cos\phi$$

$$\text{and }\tan\theta=\frac{b\cdot\sin\phi}{a+b\cdot\cos\phi}.$$

9. Resolution of Vectors.—The projection of a vector upon a line is called the component or the resolved part of the vector

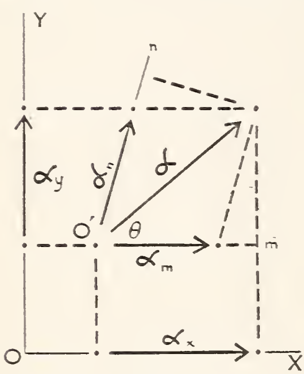

FIG. 9.

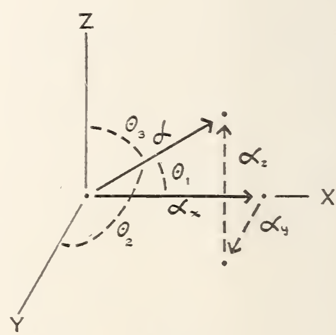

FIG. 10.

along that line. For example, in Fig. 9, α_x and α_y are the projections of α along the x- and y-axes respectively, and are called the x- and y-components of α.

$$a_x=a\cos\theta,$$

$$a_y=a\sin\theta.$$

There is a distinction between the component and resolved part of a vector which it is important to understand clearly. A resolved part is always a component but the reverse is not the case. For example, let us draw the axes $O'm$ and $O'n$ through O'. making $O'm$ parallel to OX but $O'n$ oblique to OY. Then the components of α along these axes are α_m and α_n respectively. However, these are not resolved parts. The resolved part of α along $O'm$ is α_x just as before (since $O'm$ is parallel to OX) while the resolved part of α along $O'n$ is the projection of α along that line.

In brief, resolved part is the projection of the vector along a line and it is not equal to the component along that line unless the angle between the components is a right angle.

In three dimensions the projections are illustrated in Fig. 10, where the components along the three axes are α_x, α_y, and α_z. From this we secure the vector equation

$$\alpha = \alpha_x + \alpha_y + \alpha_z$$

and the projections or components:

$$a_x = a \cos \theta_1,$$
$$a_y = a \cos \theta_2,$$
$$a_z = a \cos \theta_3.$$

When $\theta_3 = 90°$ then $\theta_1 = \theta$ and $\theta_2 = 90 - \theta$ and we have the case of the plane; as in Fig. 9,

$$a_x = a \cos \theta,$$
$$a_y = a \sin \theta.$$

If we have several vectors in space and denote by ρ_x, ρ_y and ρ_z the sum of the components along the x-, y-, and z-axes respectively we have

$$r = \sqrt{r_x^2 + r_y^2 + r_z^2},$$

where r equals the length of the resultant ρ. The angles which ρ

makes with the three coordinate axes are determined by the equations

$$\cos\theta_1 = \frac{r_x}{r}, \qquad \cos\theta_2 = \frac{r_y}{r}, \qquad \cos\theta_3 = \frac{r_z}{r},$$

where θ_1, θ_2, and θ_3 represent the angles which ρ makes with the x-, y-, and z-axes respectively.

In case all the vectors lie in one plane, say the xy plane, we have $r_z=0$, $\theta_3=90°$, $\theta_1=\theta$, $\theta_2=90°-\theta$ and

$$r=\sqrt{r_x^2+r_y^2},$$

$$\cos\theta = \frac{r_x}{r}, \qquad \sin\theta = \frac{r_y}{r}, \qquad \text{or } \tan\theta = \frac{r_y}{r_x}.$$

10. *Illustrative Example*

An automobile travels 10 miles E 30° S, then 20 miles W, then 30 miles N 45° E, then 5 miles N 30° W. Find its final position.

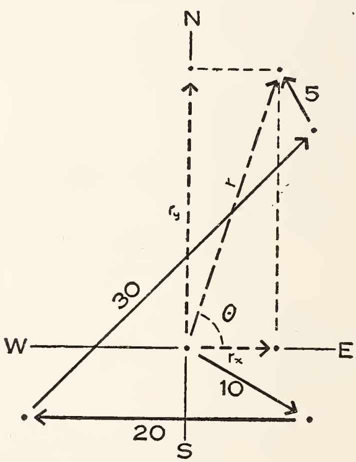

FIG. 11.

$$
\begin{aligned}
r_x &= 10 \cos 30° + 20 \cos 180° + 30 \cos 45° + 5 \cos 120° \\
&= 5\sqrt{3} - 20 + 15\sqrt{2} - 2.5 \\
&= 7.37 \text{ miles}; \\
r_y &= -10 \cos 60° + 20 \cos 90° + 30 \cos 45° + 5 \cos 30° \\
&= -5 + 0 + 15\sqrt{2} + 2.5\sqrt{3} \\
&= 20.54 \text{ miles}; \\
r &= \sqrt{7.37^2 + 20.54^2} = 21.8 \text{ miles};
\end{aligned}
$$

$$\tan \theta = \frac{r_y}{r_x} = \frac{20\ 54}{7.37} \qquad \theta = 70°16'.$$

Problems

1. Take any four vectors and draw all the vector polygons possible and show graphically that all the polygons give the same resultant vector.

2. Find the resultant vector of the two vectors whose magnitudes are 5 and 3 respectively and which make an angle of 60° with one another.

3. Show that the complex numbers $3+4i$ and $2+3i$ can be represented as vectors acting from a common point, which is the origin of real and imaginary numbers. Show that their sum obeys the parallelogram law.

4. Show that the sum of the vectors represented by the medians of any triangle, all drawn from the vertices to the opposite sides, is zero.

5. $ABCD$ is a parallelogram and Y the intersection of the diagonals, O is any point within or without the plane of the parallelogram. Show that the resultant of the vectors OA, OB, OC, OD is $4\ OY$.

6. Vectors are represented in magnitude and direction by lines drawn from any point O to the 8 corners of a cube. Show that their sum is represented by $8\ OY$ when Y is the center of the cube.

7. A man walks 4 miles north, then 6 miles northeast, then 3 miles east, then 8 miles south. Determine by diagram his distance and direction from the starting point.

8. If $ABCD$ is a quadrilateral, prove that the sum of the vectors AB, AD, CB, and CD is represented by four times the line joining the middle points of the diagonals.

9. The magnitudes and direction angles of certain vectors in space are given below. Find the resultant in magnitude and direction in each case.

(a) $10, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{5}{6}\pi; 5, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{3}{4}\pi.$
(b) $6, \frac{1}{4}\pi, \frac{2}{3}\pi, \frac{2}{3}\pi; 4, \frac{1}{6}\pi, \frac{1}{2}\pi, \frac{2}{3}\pi.$

* 10. Given the vectors $a\ (3, -2)$, $b(5, 0)$, $c(-10, 6)$, $d(7, 7)$. Construct the figures and find the resultant of the following

(a) $a+2b-3c$;
(b) $2a-b+c+2d$;
(c) $3a+4c-d$;
(d) $2a-3b-2c-2d$.

11. A point undergoes three displacements of 1, 2, and 3 units, respectively, in directions parallel to the sides of an equilateral triangle taken in order. What is the resultant displacement?

* 12. Find the resultant of three coplanar vectors a, b, c, whose components are $(3, -2)$, $(2, 6)$, $(-7, -1)$, respectively.

* 13. Given in the xy plane a vector whose axial components are $(-2, 1)$. Find its resolved part along the directed line from the origin to the point $(2, 1)$.

14. A vector drawn east has a length of 20 inches, and one drawn northeast a length of 30 inches. What is their vector sum?

* 15. Find the resultant of three coplanar vectors a, b, c, whose components are $(4, -1)$, $(-3, 8)$, $(-7, -5)$ respectively.

* (a, b) denotes a vector drawn from the origin to the point (a, b).

CHAPTER II

STATICS OF A PARTICLE

11. Statics is that portion of Mechanics which treats of bodies at rest. The treatment of bodies which are in uniform motion is closely allied to the treatment of bodies at rest, but we will restrict ourselves in the present chapter to bodies which are actually at rest.

In Mechanics the word particle is used to designate a portion of matter whose dimensions are negligible. Practically, however, a great many bodies whose dimensions are not negligible can be treated as if this were true. The body is considered as concentrated at a point and hence treated as if it were a particle.

12. Force.—*When the action of two bodies upon each other tends to produce a change in the motion of the bodies, we say that the bodies are exerting force upon each other.* Thus, if a man pushes against a solid wall he tends to move the wall in the direction in which he is pushing. Force is then the tendency to produce motion and we recognize it as a push or a pull. It has magnitude, line of action, sense and point of application. All forces are distributed, that is, act over an appreciable area or volume, but frequently this is relatively small and can be regarded as a point. Forces are considered then as concentrated forces acting at a particular point.

13. Units of Force.—The most common example of force is that due to the attraction of the earth upon a body. This force is called weight. It varies slightly for different latitudes and elevations and would vary considerably at great distances from the surface of the earth. Ordinarily no account need be taken of this variation at different places, for errors introduced into

engineering calculation, because of it, are practically always negligible.

The unit of force is the weight of a given quantity of matter as specified by law. This unit is the pound in most English speaking countries and the kilogram elsewhere. The kilogram is 2.20461 pounds avoirdupois. Both units will be used in this text.

Whenever the ton is mentioned it is to be understood, unless otherwise specified, that we mean the so-called "long ton" which is 2240 pounds.

14. Forces as Vectors; Parallelogram of Forces.—A force which has a given point of application is fully determined by its magnitude and direction. Let us take two forces A and B acting at a common point; draw a vector α whose direction is the same as that of A and whose length represents the magnitude of A. Draw also a vector β in the same manner for B. Let γ represent the resultant $\alpha+\beta$ of these two vectors. It has been verified by experiment that γ represents in magnitude and direction that force which has the same effect as the two forces A and B acting simultaneously; in other words, *the resultant γ represents the resultant of A and B. We shall therefore assume a parallelogram law of forces corresponding to the parallelogram method for vectors.*

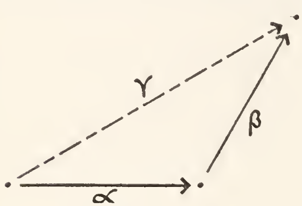

FIG. 12.

15. Polygon of System of Concurrent Forces.—Forces which have a common point of application are called concurrent forces. *If any number of forces are concurrent the resultant of the system is a force which is represented by the resultant vector obtained by considering the forces as vectors and drawing the*

vector polygon. Thus, to find the resultant of the four forces F_1, F_2, F_3, F_4, acting at the point O, we draw the polygon $O'F'_1F'_2F'_3F'_4$ whose sides are parallel and equal to the four vectors representing the forces F_1, F_2, F_3, F_4.

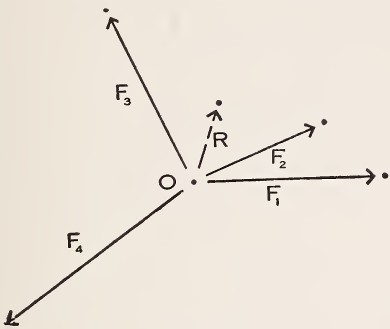

FIG. 13.

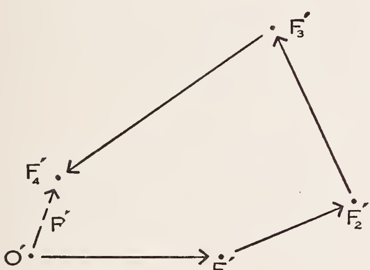

FIG. 14.

By drawing $O'F'_4$, we complete the polygon. This is the vector resultant; hence, the resultant of the four forces is R, parallel and equal to $O'F'_4$. This vector polygon is usually called the force polygon. We can draw it in various ways, since the vectors may be combined in any order, giving the same resultant. The

resultant of a system of concurrent forces is then a unique force of definite direction and magnitude, acting at the common point.

10. Equilibrium.—A particle is in equilibrium when all the forces which act upon it balance each other. It is evident that the resultant of these forces is zero, for if the resultant were of any magnitude the particle would move in the direction of this resultant. If the resultant is zero the force polygon closes. This condition enables us to calculate forces which are in equilibrium, for we can either construct the force polygon graphically and measure the unknown forces, or calculate their size by the geometrical relations of the figure. However, we can draw this force polygon only when we have not more than two unknown conditions; that is, two unknown magnitudes of forces, or two unknown directions, or a single force with both magnitude and direction unknown. It is well to note that at least two forces are required for equilibrium. A special case of the force polygon, which is of great importance, is the case of three forces in equilibrium. The force polygon is a triangle and the unknown parts of this triangle can be calculated by the geometrical or trigonometrical relations of the figure.

17. *Illustrative Examples*

Example 1.—A particle is acted upon by two forces of 10 pounds and 20 pounds respectively, making an angle of 60° with

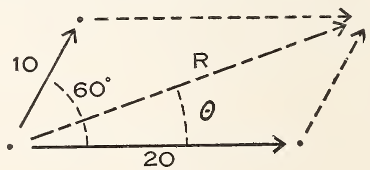

FIG. 15.

each other. Find the magnitude of the resultant and the angle which it makes with the 20 pound force.

Solution: Using the formulæ of Article 8 we have

$$R^2 = 10^2 + 20^2 + 2 \cdot 10 \cdot 20 \cos 60°,$$
$$R^2 = 700. \quad \therefore R = 26.45,$$
$$\tan \theta = \frac{10 \sin 60°}{20 + 10 \cos 60°} = \frac{8.66}{25} = .3464.$$
$$\therefore \theta = 19°6'.$$

NOTE.—On Polyphase Slide Rule set 8.66 on C over 25 on D and read 19° 6′ (approximately) on T.

Example 2.—Given a body of weight 10 pounds suspended from the ceiling by a string. The weight is pulled aside by a horizontal force F until the string makes an angle of 60° with the ceiling. Find F and the pull in the string.

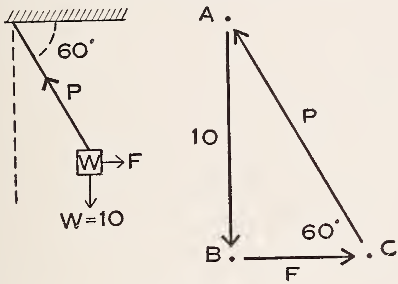

FIG. 16.

Solution: The weight 10 pounds acts downward. The body is in equilibrium under the action of three forces: W downward, F horizontal, and P the tension in the string. Draw the force polygon, which in this case is the triangle ABC. W acts vertically with a length of 10 units; F acts horizontally but is of unknown length; P makes an angle of 60° with F.

$$\therefore F = 10 \cot 60° = \tfrac{10}{3}\sqrt{3} = 5.77 \text{ lbs.}$$

NOTE.—On Slide Rule set 30° on T scale, read result on C over 10 on D scale.

$$\therefore P = 10 \csc 60° = 10 \frac{2}{\sqrt{3}} = 11.54 \text{ lbs.}$$

NOTE.—On Slide Rule set 60° on S and read result on A scale over 10 on B scale.

Example 3.—A weight of 60 pounds is suspended by two strings of lengths 12 and 9 feet respectively; the other ends of the strings are attached to two points in a horizontal line 15 feet apart; find the pull in the strings.

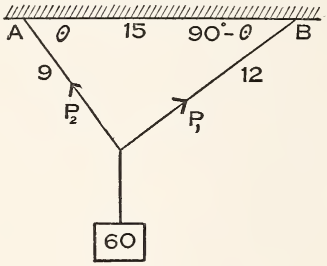

FIG. 17.

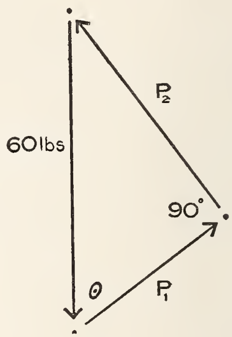

FIG. 18.

Let P_1 and P_2 represent the pull in the 12- and 9-foot strings respectively. The strings and the line AB form a right triangle.

From the force triangle (Fig. 18)

$$P_1 = 60 \cos\theta = 60 \cdot \frac{9}{15}, \qquad P_2 = 60 \sin\theta = 60 \cdot \frac{12}{15},$$

$$P_1 = 36 \text{ lbs.} \qquad P_2 = 48 \text{ lbs.}$$

Problems

16. Find the magnitude and direction of the resultant of forces equal to 5 and 12 pounds respectively, acting at right angles.

17. Forces equal respectively to 3, 4, 5, and 6 pounds act on a particle in directions respectively north, south, east and west. Find the direction and magnitude of their resultant.

18. Two forces whose magnitudes are P and $P\sqrt{2}$ pounds act on a particle in directions inclined at an angle of 135° to each other. Find the magnitude and direction of the resultant.

19. Find the resultant of two forces of 13 and 11 pounds respectively, acting at an angle whose tangent is $\frac{12}{5}$.

20. Two forces acting at an angle of 60° have a resultant equal to $2\sqrt{3}$ pounds; if one of the forces is 2 pounds, find the other force.

21. Find the resultant of two forces of 10 and 9 pounds respectively, acting at an angle whose tangent is $\frac{3}{4}$.

22. Two equal forces act on a particle. Find the angle between them when the square of their resultant is equal to three times their product.

23. An 8-pound force and an unknown force act at a point, and have a resultant of 12 pounds, which makes an angle of 30° with the 8-pound force. Find the unknown force to hundredths of a pound.

24. The resultant of two forces of 8 and 10 pounds respectively is 5 pounds. Find, to the nearest minute, the angle between the 8- and 10-pound forces.

25. The resultant of two forces P and Q acting at right angles is R. If P be increased by 9 pounds and Q by 5 pounds, R becomes three times its former value, and makes the same angle with Q that it did before with P. Find P, Q, and R.

26. A weight of 20 pounds is suspended freely from a fixed point by a perfectly flexible string. Find what horizontal pull applied to the body will move it out so that the string will make an angle of 30° with the vertical, and find the pull in the string.

27. A weight of 4 pounds is suspended by a string, and is acted upon by a horizontal force. If in the position of equilibrium the tension of the string is 5 pounds, what is the horizontal force?

28. A weight of 10 tons is hanging by a chain 20 feet long. Find how much the tension in the chain is increased by the weight being pulled out by a horizontal force to a distance of 12 feet from the vertical.

29. The weight $W = 100$ pounds is supported at O by two cords OA and OB (Fig. 19). Using the "triangle of forces," determine the tension in OA, the angles between the three forces acting at O being given.

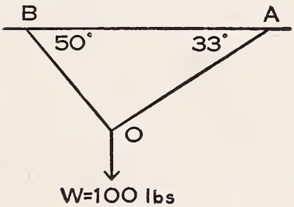

Fig. 19.

30. The sum of two forces is 18, and the resultant, whose direction is perpendicular to the lesser of the two forces, is 12; find magnitudes of the forces.

31. Three concurrent forces are represented by lines drawn from the vertices of a triangle to the middle points of the opposite sides. Show that the resultant is zero.

32. A cord whose length is $2l$ is tied to the points A and B in the same horizontal whose distance is $2a$. A smooth ring upon the cord sustains a weight W. Find the tension in the cord.

18. Types of Forces.—In the following articles we shall describe briefly some of the common forces with which we are concerned in statics, and state the assumptions made regarding them. Forces thus considered are the weight of a body, the reaction of a surface, the reaction of pins on a member of a structure and the tension in cords and cables.

The weight of a body is the force which pulls the body toward the earth. It acts in a vertical line.

19. Action and Reaction.—In a chapter on the Dynamics of a Particle, which follows later, Newton's laws of motion are stated and discussed. However, it is necessary that we should at this time understand and employ his Third Law; *i. e.,* to every

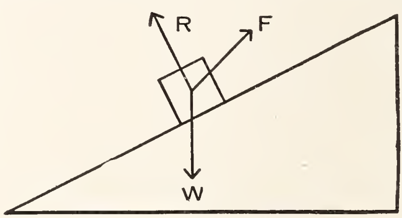

FIG. 20.

action there is always an equal and opposite reaction: or, the mutual actions of any two bodies are always equal and oppositely directed. *Thus, if one body pushes or pulls another it is itself pushed or pulled by this other with an equal and opposite force.* This law is a statement of fact, and has been verified by comparison of theory with observed results.

Reaction of Surfaces.—If a body rests upon a surface it presses against the surface because of its weight; and the surface reacts against the body. If the surface is smooth the reaction is normal to the surface at the point of contact. Thus, if a body rests on a smooth inclined plane (Fig. 20), the reaction is R, the weight W acts vertically downward. (Since the plane is smooth, some other force, as F, is necessary to keep the body from sliding.)

The student must have clearly in mind that this reaction is generally not equal to the weight.

20. Law of Transmissibility.—*If a force be applied to a rigid body at any point, the effect of this force will remain unchanged,*

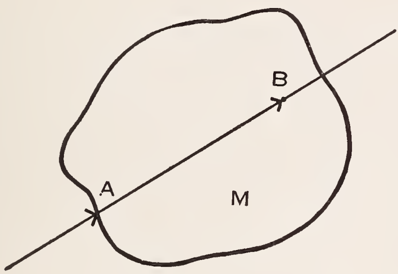

FIG. 21.

if its point of application be moved to any other position along the line of action of the force. Thus, in the figure AB is the line of action of the force F acting on the body M. No matter whether F is applied at A or B or any other point along the line AB the effect of F will be the same. This is one of the most basic facts of mechanics.

Reaction of Pins.—Many structures, such as bridges and roof trusses, are made of bars which are connected by means of pins. We shall assume for the present that these bars, or members as

they are often called, are rigid bodies (that is, do not change their shape) and that they are without weight. If the bar A is pulled by a force applied at B and acting along the axis of the bar, the pin C reacts against this force with pull equal in magnitude but opposite in direction. The bar in this case is said to be in tension, and the force acting on the bar is called the tension in the bar. Likewise, if a push be exerted at B, there must be an equal and opposite push at C. In this case the bar is said to be

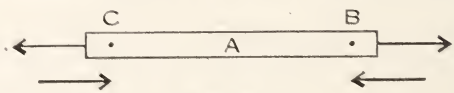

FIG. 22.

in compression and the force is called the compression. This is an illustration of the law of transmissibility of force.

Tension in Cords or Cables.—A flexible cord is one that may be bent without applying force. A cord in statics will be considered to be flexible, inextensible and weightless. For a cord to be in equilibrium every point in it must be under the influence of two equal and opposite pulls or forces. Since a cord requires no force to bend it, the direction of the tension along the cord is always parallel to the direction of the cord. Cords may carry tension in different directions by means of pulleys or pegs.

21. *Illustrative Examples*

Example 1.—Let AOB (Fig. 23) be a hinge fastened at the ends A and B, having a force P acting vertically at the apex O.

It is required to find the compression in the arms AO and BO. Drawing the force polygon (Fig. 24), we have the triangle $O'A'B'$, which is isosceles.

Hence $O'A' = O'B' = \frac{P}{2} \cdot \sec\theta$. It is interesting to note that when the hinges are nearly horizontal or θ is near $90°$ the

force in the two arms is extremely large. This mechanism is called a toggle joint and is used to obtain great pressure; an

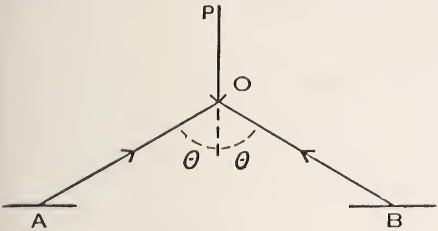

FIG. 23.

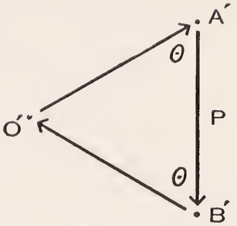

FIG. 24.

illustration of its use is found in mechanisms used in the crushing of rock.

Example 2.—Let BO be a bar of length 4 feet and AB and AO be of 5 feet and 6 feet, respectively, with a load of a hundred pounds acting at the point O. The bar BO is kept from turning by a cable AO. The problem is to find the tension in the cable and the compression in the bar.

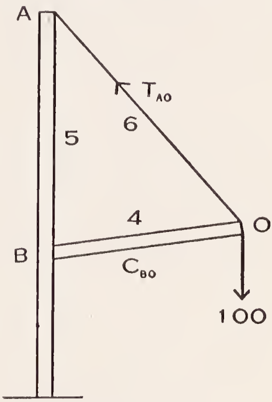

FIG. 25.

Should we now draw a force polygon (which in this case would be a triangle) it would necessarily be similar to the given triangle ABO, since its sides would be respectively parallel to the sides of ABO. We can use any unit of length whatever to represent a unit of force; hence, we can use the length 5 or AB to represent the force of 100 pounds acting vertically. In other words, the triangle ABO can be used as the force triangle; therefore we have

$$\frac{100}{5}=\frac{T_{AO}}{6}=\frac{C_{BO}}{4}.$$

Hence, $T_{AO}=120$ pounds, and $C_{BO}=80$ pounds.

NOTE.—Set 100 on C over 5 on D, on C read T_{AO} over 6 and C_{BO} over 4.

Example 3.—Suppose that a block of weight W rests on a smooth inclined plane, and we wish to find the force P necessary

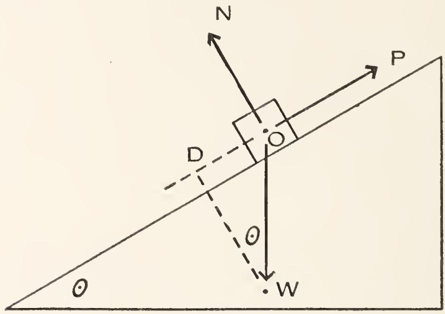

FIG. 26.

to keep it from sliding, together with the normal reaction N. Constructing the force polygon ODW, which is a right triangle, we have

$$DO = P = W \cdot \sin\theta \text{ and } DW = N = W \cdot \cos\theta.$$

Example 4.—A string BAC passes over two smooth pegs B and C and carries weights $W_1 = 4$ and $W_2 = 3$ at the ends, as shown in the figure. What weight, P, suspended from a point A, on the string, will make $\angle BAC = 120°$? Draw the force triangle HDE, and we have

$$P^2 = 4^2 + 3^2 - 2 \cdot 4 \cdot 3 \cos 60°.$$
$$\therefore P = 3.61 \text{ lbs.}$$

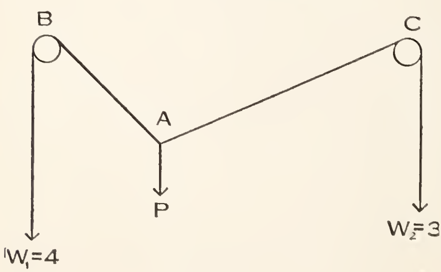

FIG. 27.

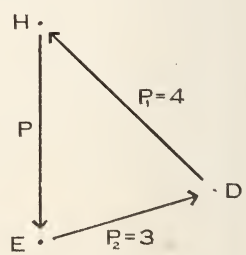

FIG. 28.

Example 5.—Let BO be a bar with a load of 100 pounds acting at the point O. The bar is kept from turning by a cable AO. To find the tension in AO, and the compression in BO. It will be seen that the triangle AOB is the force triangle, with the side AB corresponding to the 100-pound weight. We have then

$$\frac{100}{\sin 15^\circ}=\frac{T_{AO}}{\sin 30^\circ}=\frac{C_{BO}}{\sin 135^\circ}.$$

Hence,

$$T_{AO}=193 \text{ lbs.}$$

and

$$C_{BO}=273 \text{ lbs.}$$

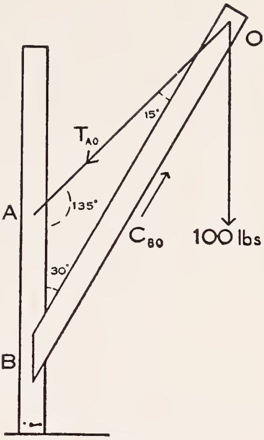

FIG. 29.

NOTE.—Set 100 on A over 15° on S and on A read 193 over 30° and 273 over 45°.

Problems

33. A weightless rod AC hinged at one end A so as to be free

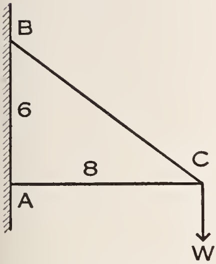

FIG. 30.

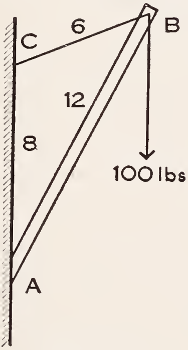

FIG. 31.

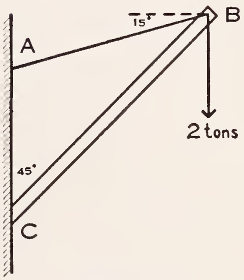

FIG. 32.

to turn in a vertical plane, is held in a horizontal position by means of the chain BC (Fig. 30). If a weight W be suspended

at C, find the thrust P in AC and the tension T of the chain. Assume AC equals 8 feet and AB equals 6 feet.

34. Find the compression in the bar AB (Fig. 31) and the tension in the cord BC.

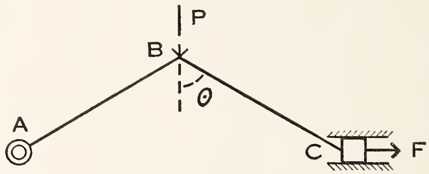

FIG. 33.

35. Find the tension in the cord AB and the compression in the bar BC, due to a load of 2 tons at B (Fig. 32).

36. If a toggle joint ABC has the end A hinged, find the horizontal force F in terms of the load P at the apex (Fig. 33).

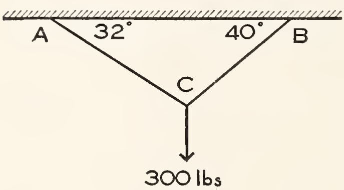

FIG. 34.

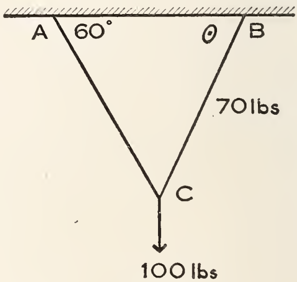

FIG. 35.

37. The 300-pound weight of Fig. 34 is attached at C to the cords BC and AC. Find the tension in BC.

38. The 100-pound weight in the figure is attached at C to the cords BC and AC. If the tension in BC is 70 pounds find the angle θ to the nearest tenth of a degree (Fig. 35).

39. A cord is attached to a fixed point and passes over a smooth pulley on the same level with the point and 10 feet distant (Fig.

36). A smooth ring of weight w slides on the cord; the cord is kept tight by a weight W on the free end. Find a depth to which the ring will sink when (1) $w=W$; (2) when $w=\frac{W}{3}$.

40. Three pegs form an equilateral triangle; if an india-rubber band be stretched around the pegs, and if the pull in the band be equal to a weight of 10 pounds, find the resultant pressure on each peg.

41. A wheel weighing W pounds is about to roll over an obstruction. The radius of wheel is R and the height of the obstruction is h. Find the horizontal force through the center necessary to start the wheel (Fig. 37).

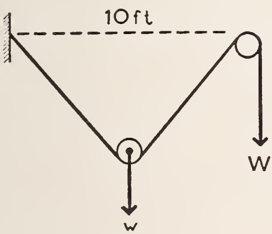

FIG. 36.

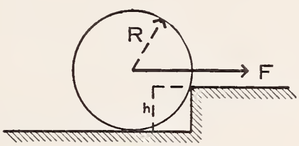

FIG. 37.

42. Two rafters making an angle of 120° support 112 pounds at the peak. Find the compressive force on each rafter.

43. Show that if a picture is hung from a nail by a cord secured to two rings in the top of the frame, the shorter the cord the stronger it ought to be. Would it be possible for the cord to remain straight when placed over the nail?

22. Resolution of Forces.—We have seen that a number of forces may be replaced by a single force. The converse of this is also true; we can replace a force by a number of forces. While the former case is unique in solution, there being only one resultant, the latter is not unique as there are an infinite number of ways a force can be resolved into component forces. The most important case of resolution of forces is that which gives us com-

ponents along two mutually perpendicular lines. Let AB be a force, then by the parallelogram law it is the resultant of the two perpendicular forces AC and AD.

Hence

$$AC = AB\cos\theta,$$
$$AD = BC = AB\sin\theta = AB\cos\phi.$$

We have then the rule: The component of a force along one of two perpendicular lines is the product of the force and the cosine of the angle between the line of action of the force and the given line.

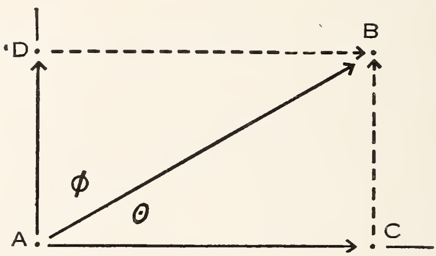

FIG. 38.

23. Algebraic Method for Finding the Resultant of a System of Concurrent, Coplanar Forces.—If a number of coplanar forces act upon a particle then they can all be resolved along any pair of mutually perpendicular lines, and the algebraic sums of the components will be the components of the resultant. If a system of coordinate axes x and y are taken, and X is the sum of the components along the x-axis and Y along the y-axis,

$$R\cdot\sin\phi = Y, \qquad \therefore R^2 = X^2 + Y^2 \qquad \text{and } \tan\phi = \frac{Y}{X};$$
$$R\cdot\cos\phi = X, \qquad R = X\sec\phi = Y\csc\phi,$$

where ϕ is the angle which the resultant R makes with the x-axis.

In calculating the resultant due regard must be taken of the signs of the trigonometric functions. It might also be noted that a force equal and opposite to R would put the system in equilibrium; this balancing force is called the equilibrant.

Example.—Let us consider seven coplanar forces acting on a particle, as follows: $\frac{3}{2}\sqrt{3}$, $4\sqrt{2}$, 2, 3, 6, $5\frac{1}{2}$, and 8 pounds, making with a fixed line in the plane these angles: 0°, 45°, 120°, 150°, 240°, 270°, and 300°, respectively. Required the resultant force and the angle it makes with the x-axis.

$$\begin{aligned}
\tfrac{3}{2}\sqrt{3}\cos 0^\circ &= \tfrac{3}{2}\sqrt{3}\\
4\sqrt{2}\cos 45^\circ &= 4\\
2\cos 120^\circ &= -1\\
3\cos 150^\circ &= -\tfrac{3}{2}\sqrt{3}\\
6\cos 240^\circ &= -3\\
5\tfrac{1}{2}\cos 270^\circ &= 0\\
8\cos 300^\circ &= 4\\
\hline
X &= 4
\end{aligned}$$

$$\begin{aligned}
\tfrac{3}{2}\sqrt{3}\sin 0^\circ &= 0\\
4\sqrt{2}\sin 45^\circ &= 4\\
2\sin 120^\circ &= \sqrt{3}\\
3\sin 150^\circ &= \tfrac{3}{2}\\
6\sin 240^\circ &= -3\sqrt{3}\\
5\tfrac{1}{2}\sin 270^\circ &= -5\tfrac{1}{2}\\
8\sin 300^\circ &= -4\sqrt{3}\\
\hline
Y &= -6\sqrt{3}
\end{aligned}$$

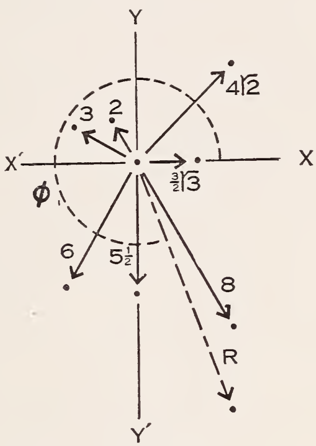

FIG. 39.

$$R=\sqrt{X^2+Y^2}=\sqrt{16+108}=2\sqrt{31}=11.14,$$

$$\phi=\arctan\frac{Y}{X}=\arctan-\tfrac{3}{2}\sqrt{3},$$

$$\phi=291^\circ 3'.$$

NOTE.—To find 8 cos 300° set 30° on S and read result on B under 8 on A; to find 8 sin 300°, set 60° on S and read result on B under 8 on A, etc.

Set 1 on C scale under 3 on A scale, move the runner to 3 on C scale; then set 2 on C scale under runner, read 21° 3′ on T scale.

$$\therefore \phi = 270^\circ + 21^\circ\ 3' = 291^\circ\ 3'.$$

Set 21° 3′ on S and read 11.2 on A scale over 4 on B scale.

24. Algebraic Conditions for Equilibrium.—If a system of forces acting upon a particle is in equilibrium, the resultant is zero. If the resultant is zero its resolved part along any line must be zero. We have then, for a system of coplanar forces in equilibrium, the following important law: *The algebraic sum of the components along any line must equal zero.*

Since we can resolve a force along any line, it is well to select the line (or lines) in such a way as will tend to simplify the algebraic work. This may often be done by resolving in a direction perpendicular to one of the unknown forces.

Example 1.—Given the forces shown in Fig. 40, acting on a body W, weighing a hundred pounds, and resting on the smooth inclined plane AB. Find the force, P, just sufficient to keep

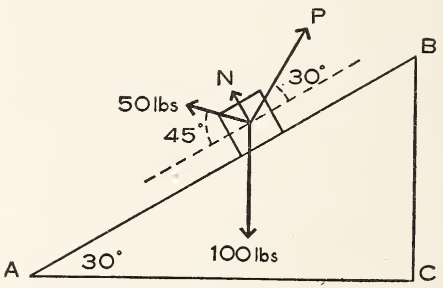

FIG. 40.

the body from sliding; also the normal pressure or reaction N. Let us resolve along the plane and perpendicular to it.

We have then

$$X = 0 = P \cdot \cos 30° - 50 \cdot \cos 45° - 100 \cdot \cos 60°;$$
$$\therefore P = 98.53 \text{ lbs.}$$

Also

$$Y = 0 = N + 50 \cdot \sin 45° + P \cdot \sin 30° - 100 \cdot \sin 60°;$$
$$\therefore N = 86.60 - 49.27 - 35.36 = 1.97 \text{ lbs.}$$

In solving for N, we might have resolved perpendicular to P, and thus secured an equation in N alone; but this would have proved less simple than the method used.

Often we have problems in which it is necessary to solve successively for the various unknowns. If a series of particles or joints are connected by a system of ropes or bars, we may start our solution at one of the joints. The joint must be such that not more than two of the forces acting on it are unknown. This joint is in equilibrium, and hence we can solve for the unknown

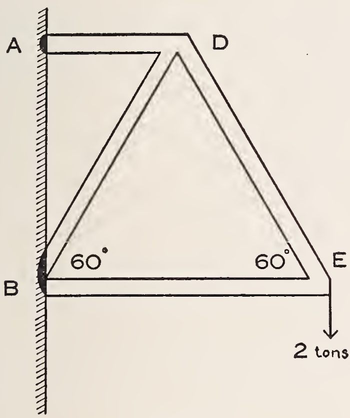

FIG. 41.

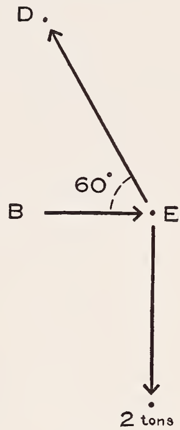

FIG. 42.

forces acting upon it. This will give us known forces and will usually enable us to proceed to a neighboring point and solve its system. It must be kept constantly in mind that every joint is in equilibrium under the forces which act upon it and that the force exerted at one end of a member is equal and opposite to that exerted at its other end.

Example 2.—Let Fig. 41 represent a crane, supporting a load, at the point E, of two tons. To find the stresses in the members, start at the point E. The forces acting at this point are the weight, and the stresses in the two members BE and ED. The stress in ED must evidently act *upwards* along ED, to balance the downward pull of the load, since the stress in BE has no

vertical component. Similarly, the stress in BE must act upon E towards the right. Resolving horizontally and vertically, we have

$$X=0=BE-ED\cdot\cos 60°,$$
$$Y=0=ED\cdot\sin 60°-2;$$
$$\therefore ED=\frac{4}{\sqrt{3}} \quad \text{(tension)},$$
$$BE=\frac{2}{\sqrt{3}} \quad \text{(compression)}.$$

Note that in the case of tension the stress pulls on the pin or joint at E; whereas in the case of compression the stress pushes against the point E.

Let us now proceed to the point D. Since the member DE is in equilibrium the force acting at D must be equal and opposite

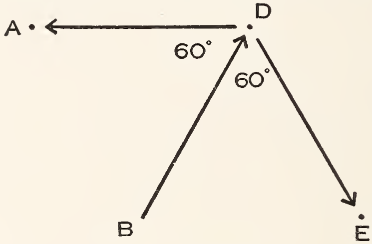

FIG. 43.

in direction to the force acting at E. Thus, the forces acting at D are shown in Fig. 43. Resolving horizontally and vertically, as above, we get

$$X=0=-AD+BD\cdot\cos 60°+\frac{4}{\sqrt{3}}\cos 60°,$$

or

$$AD-\frac{BD}{2}=\frac{2}{\sqrt{3}}; \qquad (1)$$

also

$$Y=0=BD\cdot\sin 60°-\frac{4}{\sqrt{3}}\sin 60°,$$
$$\therefore BD=\frac{4}{\sqrt{3}} \quad \text{(compression)};$$

and from (1),

$$AD=\frac{4}{\sqrt{3}} \quad \text{(tension)}.$$

Problems

44. Four forces of 8, 26, 20, and 16 pounds act on a particle; the angles between the 8-pound force and the other forces are respectively 20°, 65°, and 70°. Find graphically the resultant. (Let 1 inch represent a force of a 4-pound weight.)

45. Forces of 11, 16, 13, and 10 pounds act at a point O in the plane XOY, and make angles of 30°, 90°, 180°, and 310°, respectively, with OX. Find the resultant of these four forces to hundredths of a pound and the angle (to the nearest tenth of a degree) which the resultant makes with OX.

46. A captive balloon capable of raising a weight of 400 pounds, is anchored at a height of 400 feet by a rope 500 feet long. Find the pull in the rope and the horizontal pressure of the wind on the balloon.

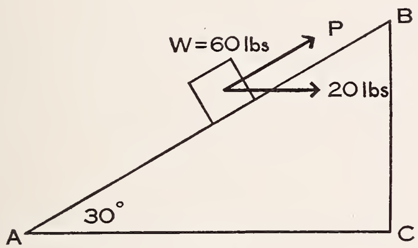

FIG. 44.

47. The tow-line of a canal boat is 150 feet long from the fastening on the boat to the harness of the mule. The tow-line includes a spring-balance which reads 90 pounds. The mule walks 3 feet from the edge of the water, and the attachment of the tow-line to the boat is 12 feet from the bank. Neglect the sag of the line.

(a) What is the force tending to turn the bow toward the bank?

(b) What is the effective pull on the boat in the direction of the desired motion?

(c) What is the percentage of the former to the total pull?

(d) How would the sag of the line modify these results?

48. W is held at rest on the smooth plane AB by the horizontal 20-pound force, shown in Fig. 44, and by the force P, and the

reaction of the plane on W. P acts parallel to AB. Find the force P, and the reaction of the plane AB on W.

49. A 160-pound weight rests on the smooth plane AB, and is held in equilibrium by two forces of 80 pounds each, acting paral-

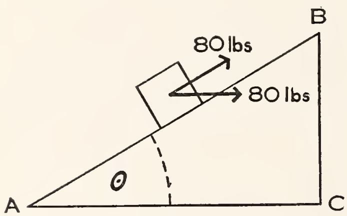

FIG. 45.

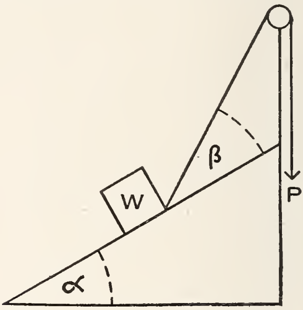

FIG. 46.

lel to AB and AC, respectively. Find the inclination of AB to the horizontal (to the nearest min.) and the reaction of AB on W (Fig. 45).

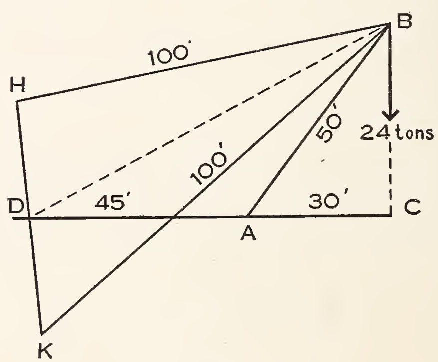

FIG. 47.

50. A weight W rests on a smooth plane and is kept in position by a string passing over a pulley and attached to a weight of P pounds hanging freely (Fig. 46). Find all the forces when

(a) $\alpha=30°$, $\beta=30°$, $P=15$ lbs.
(b) $\alpha=40°$, $\beta=15°$, $W=10$ lbs.

51. The 50-foot spar AB, used as a derrick, is supported by two back stays, BH and BK, each 100 feet long. Given $DA=45$ feet, $AC=30$ feet (Fig. 47), find the tension in each back stay when a weight of 24 tons hangs at B (neglect weight of spar).

25. Friction.—When we attempt to slide one body over another there is resistance offered to the motion because of the roughness of the two bodies. This resistance is called friction. There are three fundamental laws of friction with which we shall be concerned:

(1) The friction between two bodies is directly proportional to the pressure.

(2) The amount of friction for any given pressure is independent of the area of contact.

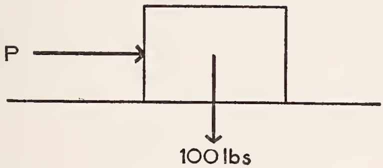

FIG. 48.

(3) The direction of the friction is tangent to the surface upon which a body tends to move and is opposite to the direction in which the body tends to move.

These laws do not apply when the pressure is so great as to cut one of the sliding bodies. The ratio of the friction to the pressure, that is, to the normal pressure, is called the coefficient of friction, and is designated by the letter μ. Thus $\mu=\frac{F}{N}$.

The friction F is the total force that the roughness of the bodies can offer to sliding. It must be remembered that the fric-

tion or resistance that a body offers to movement is equal to the force applied, in the direction of motion, until the friction reaches its maximum value. Thus, if a block weighs 100 pounds and the coefficient of friction is $\frac{1}{2}$, the total force that the friction can exert is 50 pounds.

If the force P is greater than 50 pounds the block will slide. On the other hand, if the force P is only 2 pounds the friction is only 2 pounds. Ordinarily when we use the word friction we mean the maximum or so-called "limiting friction."

26. The Angle of Repose.—If a body of weight W is placed on a rough inclined plane, as in Fig. 49, the friction tends to keep the body from sliding. If the angle θ is increased until slipping just begins we have

$$\tan\phi = \frac{F}{N} = \frac{W\sin\theta}{W\cos\theta} = \tan\theta = \mu \quad \text{and} \quad \theta = \phi$$

or the tangent of this angle is equal to the coefficient of friction. This angle, which the inclined plane makes with the horizontal

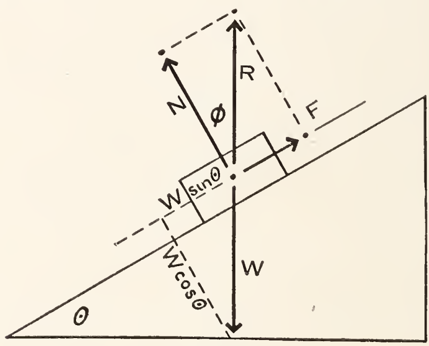

FIG. 49.

at the instant the body begins to slide, is called the *angle of repose.* The coefficient of friction is usually determined by placing a body on an inclined plane and finding the angle of repose.

When a body moves (or is just on the point of moving) on a rough surface the reaction is no longer normal to the surface but acts at an angle equal to the angle of repose from the normal; it always acts opposite to the direction of motion.

27. *Illustrative Problems on Friction*

Example 1.—A block is being acted upon by a force P which is about to slide the block along the floor.

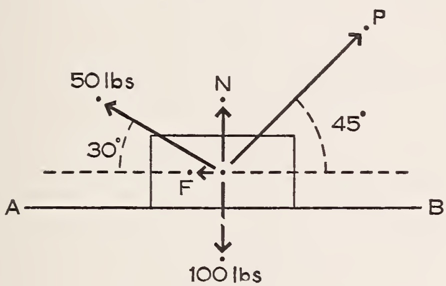

FIG. 50.

The weight of the block is 100 pounds and the coefficient of friction is $\frac{1}{4}$. There is also another force of 50 pounds pulling back on the block. To find the value of P that will just cause the block to move towards B. Let the reaction of the plane on the block be N, and the friction be F. Then $F=\frac{N}{4}$. Resolve horizontally and vertically

$$X=0=P \cdot \cos 45° - 50 \cdot \cos 30° - \frac{N}{4},$$

$$Y=0=P \cdot \sin 45° + N + 50 \cdot \sin 30° - 100;$$

$$\therefore N=25.36 \text{ lbs., and } P=70.2 \text{ lbs.}$$

Example 2.—An unknown weight W and a weight of 20 pounds rest on a double inclined plane as in the figure; the angles which the two planes make with the horizontal are 45° and 30°, respectively. The weights are connected by a string passing over a smooth pulley. The coefficient of friction between the 20-

pound weight and the plane on which it lies is $\frac{1}{6}\sqrt{3}$; the other plane is smooth. Find the weight W and the tension in the

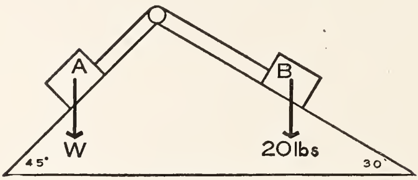

FIG. 51.

string, (a) when the 20-pound weight is about to move *up* the plane; (b) when it is about to move *down* the plane.

At the point A there are three forces acting; W, vertically; T, the tension in the string, which acts along the plane; and N, the normal reaction of the plane. This is illustrated in Fig. 52. At B, however, we have four forces acting: 20 pounds vertically, T along the plane, N' normal to the plane, and F the friction along the plane. F acts opposite to the tendency of motion; *i. e.*, (a) if the 20-pound weight is about to move *up* the plane F acts *down* the plane (Fig. 53); and (b) if the 20-pound weight is about to slide *down*, F acts *up* the plane (Fig. 54).

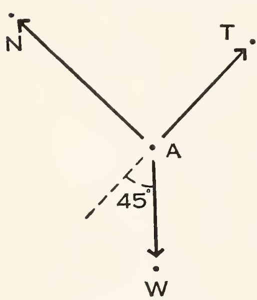

FIG. 52.

In Fig. 52, resolving perpendicular to N, we get

$$T = W \cdot \sin 45° = \frac{W}{\sqrt{2}} \tag{1}$$

In Fig. 53, resolving perpendicular to N',

$$T = F + 20 \cdot \cos 60° = F + 10;$$

and resolving along N',

$$N' = 20 \cdot \cos 30° = 10\sqrt{3}.$$

But

$$F = \mu N' = \frac{1}{2\sqrt{3}} \cdot 10\sqrt{3} = 5 \text{ lbs.}$$

$$\therefore T = 10 + 5 = 15 \text{ lbs.}$$

and from (1),

$$W = T\sqrt{2} = 15\sqrt{2} \text{ lbs.}$$

These are the values of W and T when the 20-pound weight is about to move *up* the plane. In the same way, using Fig. 54, and

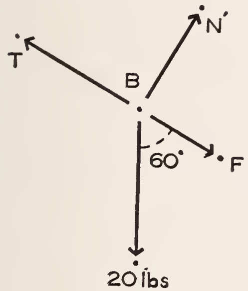

FIG. 53.

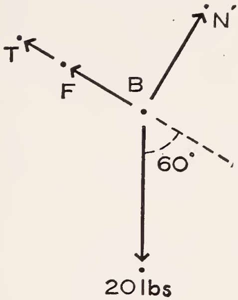

FIG. 54.

resolving perpendicular to N', we have $T+F=20 \cdot \cos 60° = 10$; $\therefore T=10-F$.

Resolving along N' . . . . $N'=10\sqrt{3}$.

Therefore

$$F=5 \text{ lbs.}$$
$$T=5 \text{ lbs.}$$

and from (1), as before,

$$W=T\sqrt{2}=5\sqrt{2} \text{ lbs.}$$

These are the values which obtain when the 20-pound weight is about to move *down* the plane.

Problems

52. A block, weight 100 pounds, resting on a horizontal plane, is pulled by a force P which makes an angle of 10° with the horizontal plane. If the coefficient of friction is 0.2, how great must P be to start the block?

53. If a block of weight W rests on a horizontal plane, find the angle that will require a minimum force, P, to just start the block in motion. The coefficient of friction is μ.

54. If in problem 52 the force P is pushing against the block, find P.

55. Solve problem 52, using 25° and 0.3 in lieu of 10° and 0.2, respectively.

56. Show that the force P (Fig. 55), inclined at an angle ϕ to the plane, that will (a) just move the weight up the plane, (b) just prevent it from sliding down the plane, is

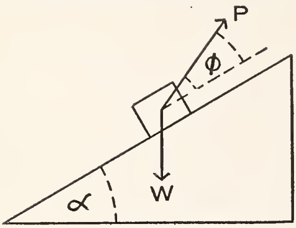

FIG. 55.

$$\text{(a)}\ P=\frac{W\sin(\alpha+\theta)}{\cos(\phi-\theta)},$$

$$\text{(b)}\ P=\frac{W\sin(\alpha-\theta)}{\cos(\phi+\theta)},$$

where θ is the angle of friction (repose).

57. Show that the least values of P in the preceding problem occur when $\phi=\theta$ in (a) and when $\phi=-\theta$ in (b); *i. e.*, when

$$\text{(a)}\ P=W\sin(\alpha+\theta), \qquad \text{(b)}\ P=W\sin(\alpha-\theta).$$

58. In Fig. 56 the weight W is raised by a horizontal force P. If the only friction is between the surfaces of the wedge and the weight, prove that the value of P just sufficient to raise the weight is

$$P=W\tan(\alpha+\theta),$$

where θ is the angle of friction.

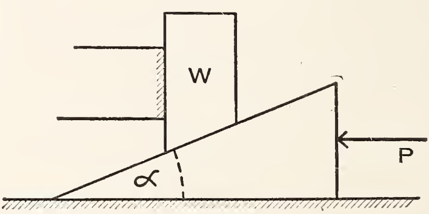

FIG. 56.

59. A body, whose weight is 20 pounds is about to slide along a rough horizontal plane, under the action of a force whose inclination to the horizontal is 45°. Find the magnitude of the force, μ being 0.3.

60. A weight W rests in equilibrium on a rough inclined plane, being just on the point of slipping down. On applying a force W,

parallel to the plane, the weight is just on the point of moving up. Find the angle of the plane and the coefficient of friction.

61. A weight of 30 pounds is resting on a rough horizontal plane and can just be moved by a force of 10 pounds acting horizontally; find the coefficient of friction and the direction and magnitude of the resultant reaction of the plane.

62. Two rough bodies W_1 and W_2 rest upon an inclined plane and are connected by a string parallel to the plane. If the coefficient of friction is not the same for both, determine the greatest inclination consistent with equlibrium, and the tension of the string.

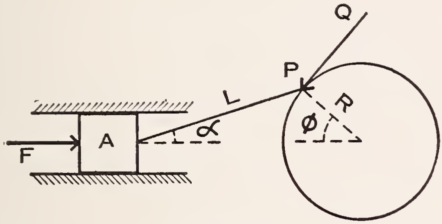

FIG. 57.

63. A force F acts upon a sliding member A (Fig. 57), which is connected to a bar AP, and this bar is connected to a fly wheel. Find the force Q, perpendicular to the radius, that will keep the wheel in equilibrium, if the coefficient of friction at A is μ.

Review Problems

64. A force equal to 10 pounds is inclined at an angle of 30° to the horizontal; find its resolved parts in a horizontal and vertical direction respectively.

65. Find the resolved part of a force P, in a direction making (1) an angle of 45°, (2) an angle equal to the $\cos^{-1} \frac{12}{13}$ with its direction.

66. A truck is at rest on a railway line, and is pulled by a horizontal force of 100 pounds, in a direction making an angle of 60° with the direction of the rails. What is the force tending to urge the truck forward?

67. Find the components of a force of 50 pounds along two directions, making angles of 60° and 45° with it on opposite sides.

68. If a force P be resolved into two forces making angles of 45° and 15° with its direction, show that the latter force is $P\sqrt{\frac{2}{3}}$.

69. If a force $F=20$ pounds be resolved into two mutually perpendicular components, one of which is double the other, find the greater component, and the angle it makes with F.

70. A force of 25 pounds acting vertically upwards is the resultant of two forces, one being horizontal and equal to 10 pounds. What are the magnitude and direction of the other force?

71. Find a horizontal force and a force inclined at an angle of 60° with the vertical whose resultant shall be a given vertical force F.

72. (a) Three forces acting at a point are in equilibrium. If they make angles of 120° with one another, show that they are equal.

(b) If the angles are 60°, 150°, and 150°, in what proportions are the forces?

73. Three forces acting on a particle are in equilibrium; the angle between the first and second is 90°, and that between the second and third is 120°; find the ratio of the forces.

74. Forces equal to $5W$, $12W$, and $13W$ acting on a particle are in equilibrium; find the angles between their directions.

75. Construct geometrically the directions of two forces $2W$ and $3W$ which make equilibrium with a force of $4W$ whose direction is given.

76. The sides AB and AC of a triangle ABC are bisected in D and E. Show that the resultant of forces, represented by BE and DC is represented in magnitude and direction by $\frac{3}{2}$ BC.

77. Two forces act at an angle of 120°. The greater is represented by 80, and the resultant is at right angles to the lesser. Find the latter.

78. If one of two forces be double the other, and the resultant be equal to the greater force, find the angle between the forces.

79. Two forces acting on a particle are at right angles and are balanced by a third force, making an angle of 150° with one of them. The greater of the two forces being 3 pounds, what must be the values of the other two?

80. AB is a 13-foot boom. A is a small hinge, and BC is a 20-foot guy (Fig. 58). Find the thrust in AB and the tension in BC, when 33 tons hangs from B.

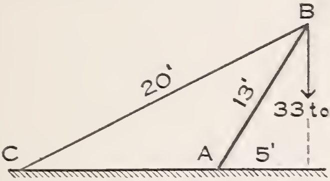

FIG. 58.

81. A weight of 42 pounds is supported by two strings of lengths 25 and 17 inches, which are tied to two pegs in the same horizontal line at a distance of 28 inches apart; find the tension in the strings, and the greatest horizontal force which can be applied to the weight without disturbing the equilibrium.

82. A weight of 270 pounds is suspended from a crane of dimensions shown in Fig. 59. It is required to find the tension in AB and the compression in BC.

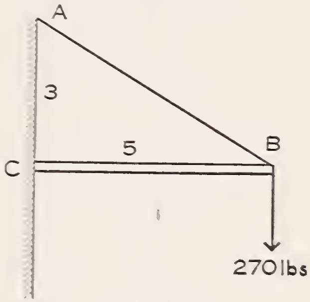

FIG. 59.

83. A small ring is held at the center of a hexagon by six cords, all in the same plane, and attached each to a separate vertex of the hexagon. The tensions of four consecutive cords are 2, 7, 9, and 6 pounds, respectively. Find the tension in the remaining two cords.

84. P, Q, and R are weights attached to strings, two pass over pulleys and the three strings are knotted together at O (see Fig. 60). Find the tension in the strings when

(a) $Q = 10$ lbs., $\alpha = 150°$, $\beta = 60°$;
(b) $P = 5$ lbs., $\alpha = 120°$, $\beta = 90°$;
(c) $R = 6$ lbs., $\alpha = 100°$, $\beta = 140°$.

85. P, Q, and R are weights attached to strings knotted at O,

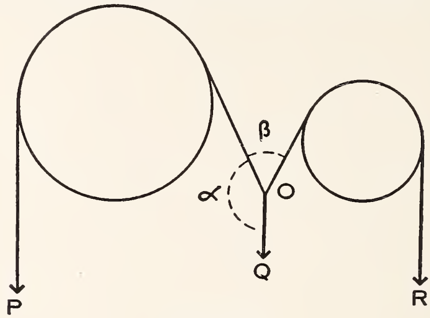

FIG. 60.

and passing over two pulleys (see Fig. 60). Find the tensions in the strings when

(a) $P=$ 15 lbs., $\alpha=90°$, $\beta=120°$;
(b) $Q=2000$ lbs., $\alpha=70°$, $\beta=150°$;
(c) $R=$ 2.5 lbs., $\alpha=60°$, $\beta=150°$.

86. On the opposite sides of a straight line AB two equilateral triangles with vertices C and D are described, and a force P acts along AB. Resolve this force along the directions AC and AD.

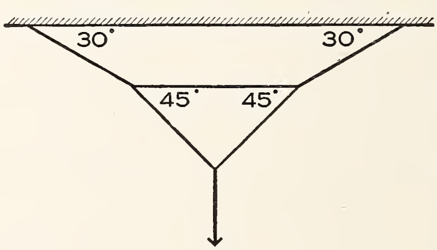

FIG. 61.

87. AD is the perpendicular from the angle A of an equilateral triangle ABC on the opposite side BC. If the resolved parts

along AC of the two forces P and Q which act along AB and AD respectively are equal to one another, find the ratio of P to Q.

88. Three strings are knotted together into the shape of an isosceles right-angled triangle (Fig. 61). A weight of 10 pounds is suspended from the right angle, and the hypotenuse is kept horizontal by two strings attached to its ends, which are both inclined at 30° to the horizon. Find the tension of each string.

89. Let R be the effective piston pressure of a steam engine and ϕ the angle between the direction of motion of the piston and the connecting rod at any moment. Show that the thrust in the connecting rod is $R \sec \phi$ and the pressure on the guide bars is $R \tan \phi$. For what position of the crank is the pressure on the guides greatest? What is the greatest pressure when the connecting rod is four times as long as the crank?

90. The force $R \sec \phi$ transmitted by the connecting rod is resolved at the crank pin tangentially and radially. Determine the tangential component which is the effective turning force and show that, if R be represented by the length of the crank, this tangential component is represented by the intercept made by the connecting rod on the radius of the crank circle which is perpendicular to the motion of the piston. Show also that when the connecting rod is long the tangential force is approximately $= R\left(\sin\theta + \frac{1}{2m}\sin 2\theta\right)$, where θ is the crank angle and m the ratio of the length of the connecting rod to that of the crank.

91. An anchor weighing 4000 pounds is supported by two tackles from the fore and main yards of a vessel, making angles of 30° and 45° respectively with the vertical. Find the tension in each tackle.

92. A barrel 4 feet long, weighing 500 pounds, is hoisted from a ship's hold by means of a pair of can-hooks 52 inches long. Find the tension on each leg of the can-hook.

93. Show that the least force which will move a weight W along a rough horizontal plane is $W \sin \phi$, where ϕ is the angle of friction.

94. $ABCD$ is a parallelogram; a particle P is attracted toward A and C by forces which are proportional to PA and PC, respectively, and repelled from B and D by forces proportional to PB and PD; show that P is in equilibrium wherever situated.

95. Two forces make an angle of 60° with each other, and their resultant is 14 pounds. If one force is 4 pounds greater than the other, find the two forces and the angle (to the nearest minute) which their resultant makes with the greater force.

96. Find the thrust on the boom CB of the Fig. 62, due to the load $W=3000$ pounds.

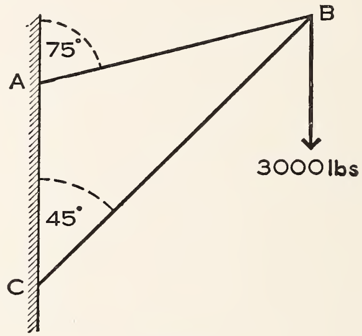

FIG. 62.

97. A weight of 30 pounds is resting on a rough horizontal plane and can be just moved by a force of 10 pounds acting horizontally; find the coefficient of friction and the direction and magnitude of the resultant reaction of the plane.

CHAPTER III

FORCES ACTING ON A RIGID BODY

28. Definition.—A rigid body is one which experiences no change of shape when acted upon by forces however large. Or, mathematically speaking, a rigid body is one any two points of which remain the same distance apart, no matter how great the forces may be that act upon it. No such bodies exist in nature, but some may be regarded as rigid so long as the forces acting are not too great.

Thus if moderate forces are applied to a cube of wood or steel, no appreciable change occurs in its size or shape, whereas, if the same forces are applied to a piece of rubber or putty, deformation takes place. Within suitable limits, then, we can regard the steel and wooden cubes as rigid bodies.

This is an illustration of a common practice in applied mathematics, namely, that of making, for the sake of simplicity, an assumption which is known to be not strictly true, but which is near enough to conformity with the facts of nature to give us results which are very nearly correct. In many cases these results serve well enough for all practical purposes. In other cases, where they are not sufficiently accurate, they furnish a first approximation, that is, a starting point for more refined investigation. Thus it is convenient and practical for many purposes to regard pieces of wood, steel, etc., as rigid bodies.

29. A rigid body can move about in space without changing the direction of any line in it. This is a motion of translation. It can turn about a fixed axis. This is a motion of rotation. It can be shown that any displacement of a rigid body is one or the other of these two or a combination of them. The body has no motion when any three points, not in the same straight line, are fixed.

For, let A, B, C, be three points not in the same straight line. If we fix A and B, the body can have only one motion, that of rotation about the line AB, since by hypothesis the body is rigid and the distance between AB and any other point P remains unchanged. This point P must describe a circle with axis AB. Thus, if C is not in line with AB, it must describe a circle about the line AB. But if C is fixed this does not happen. Hence no motion can take place and the body is fixed.

30. If a system of forces acts upon a rigid body, we frequently find that it may be replaced by a simple system or a single force without any change in its effect on the body, that is, in its tendency to produce translation or rotation. Such a single force or simplified system which is equivalent to the given system is called the resultant of the system. This use of the word "resultant" is a generalization of its use as applied to a particle in the preceding chapter. There, of course, the only possible effect was a tendency to produce translation, and the only law of substitution was found to be the parallelogram law. For a rigid body we find it sufficient to make, in addition to this, the two following assumptions:

(1) That the effect of a force on a rigid body is unchanged if its point of application is moved to any other point of the body in the line of action of the force.

(2) That the effect on a rigid body is unchanged if two equal and opposite forces, acting at any point of the body, are removed or introduced.

31. The Moment of a Force About a Line.—*Definition:* (1) If a line L is perpendicular to the line of action of a force F, the moment of the force with respect to the line L is the product of the force and the length of the common perpendicular between the two lines. Thus, in the Fig. 63, the moment of F about an axis through O and perpendicular to the plane of the paper is $F \times AO$.

(2) If the line L is not perpendicular to the line of action of the force F (Fig. 64), resolve the force into two components, one parallel and the other perpendicular to the line L. Then the moment of the force is the moment of its perpendicular component F_1 about the line L.

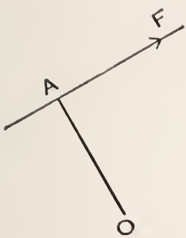

FIG. 63.

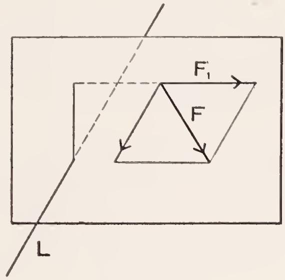

FIG. 64.

In both cases the moment measures the tendency to produce rotation of a body about the line L. In the second case the parallel component has no tendency to produce rotation about the line L. The line L is called the axis of moments. The unit of measurement for the moment of a force must involve the units of force and distance, and accordingly the unit of moment of force is the pound-foot, the ton-inch, or the like. Thus a force of 10 pounds acting perpendicular to a certain line and at a distance of 5 feet from that line has a moment of 50 pound-feet about that line as axis.

32. When we consider moments of forces in one plane only, we shall speak of the moment of a force about a point. The moment of a force about a point O is equal to the product of the magnitude of the force and the perpendicular distance from the point to the line of action of the force. This perpendicular

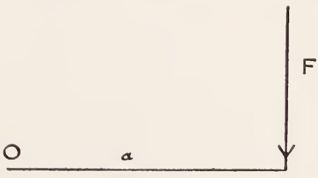

FIG. 65.

distance is known as the arm of the force. The point O is called the center of moments. The plane determined by the point O and the line of action of the force is the plane of moments. The axis of moments is a line through O, perpendicular to the plane of moments. Then, according to the definition, the moment of F about O equals $F \cdot a$ (Fig. 65).

33. Sign of a Moment.—If the tendency of the moment is to turn the body counter-clockwise, the moment is positive; if clockwise, it is negative.

34. *Theorem.*—The moment about a point O of a force F acting at a point A is equal to OA multiplied by the component of F in a direction perpendicular to OA.

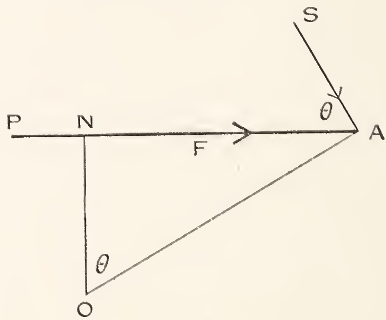

FIG. 66.

Let PA be the line of action of a force F in the plane of the paper acting on a particle at A (Fig. 66). Draw a perpendicular ON from any point O to PA. Let θ be the angle AON. Draw AS perpendicular to OA.

Then the moment of force F about $O = F \cdot ON$
$= F \cdot OA \cos\theta$
$= OA \cdot F \cos\theta$
$= OA \cdot$ component of F along SA.

35. *Theorem.*—The sum of the moments about any point O of any number of concurrent forces is equal to the moment about O of their resultant.

Let F_1, F_2, F_3 . . . . be the forces acting at A, and R their resultant. Referring to the above figure (Fig. 66), we have to show that $OA \cdot$ component of F_1 along $AS + OA \cdot$ component of F_2 along $AS +$. . . . $= OA \cdot$ component of R along AS.

Dividing through by OA we have only to show that the component of R along AS is equal to the sum of the components of F_1, F_2, F_3 . . . . along AS, which is known to be true. (See Art. 23.)

If a particle is in equilibrium under the action of any number of forces, the resultant of all these forces must be zero. The sum of the moments taken about any point whatever is equal to the moment of the resultant and is therefore zero.

Hence, when a particle is in equilibrium under the action of any number of concurrent forces, the sum of the moments of these forces about any point whatever must vanish.

36. Non-Parallel Forces.—Let P and Q (Fig. 67) be two non-parallel coplanar forces acting on a rigid body at A and B.

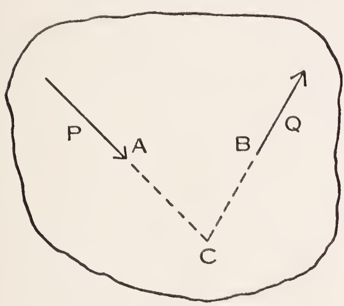

FIG. 67.

By Art. 30 these forces may be considered as acting at C, the point of intersection of their lines of action. Hence by the parallelogram law, a single force can be found, say R, to replace

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the two forces P and Q. If any number of forces act, the parallelogram law applied first to two forces, then to their resultant and another force, and so on, will give a single force as a resultant, provided we are not required, in the course of this process, to combine two parallel forces. This case will now be considered.

37. Parallel Forces.—We shall divide the consideration of parallel forces acting on a rigid body into three cases:

Case I: When the forces act in the same direction.

Case II: When the forces are unequal and act in opposite directions.

Case III: When the forces are equal and act in opposite directions.

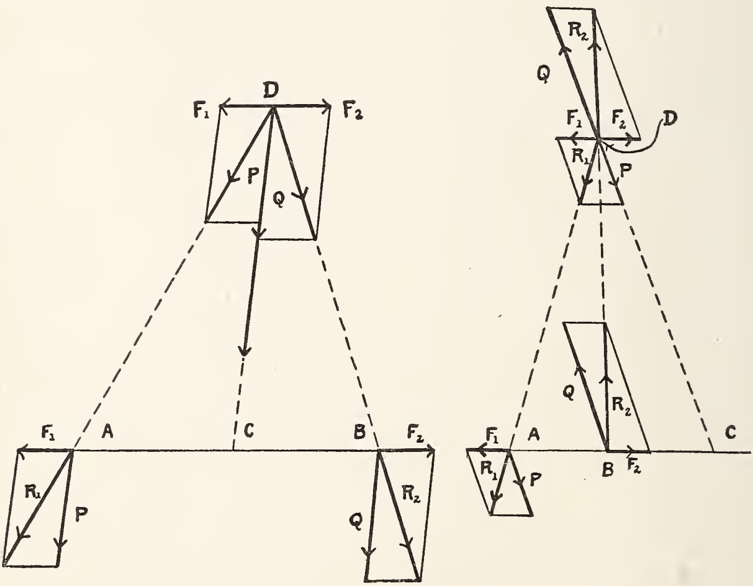


FIG. 68.

38. Cases I and II.—Let P and Q (Fig. 68), be two parallel forces acting on a rigid body and let AB be any line cutting their lines of action in A and B, respectively. At A and B in the line AB put in two equal and opposite forces F_1 and F_2. These forces have no effect on the body. Let R_1, the resultant of P and F_1, and R_2, the resultant of Q and F_2, intersect at D. R_1 and R_2 may now be regarded as acting at D. We resolve R_1 and R_2, acting at D, into components equal and parallel to their original components. F_1 and F_2 annul each other and we have left P and Q, having the same line of action at D.

In Case I, $R=P+Q$.
In Case II, $R=P-Q$.

The resultant, R, may be regarded as acting at C, the intersection of its line of action with the line AB.

From similar triangles, we have

$$\frac{AC}{DC}=\frac{F_1}{P},$$

$$\frac{CB}{DC}=\frac{F_2}{Q}.$$

Since $F_1=F_2$, dividing, we have

$$\frac{AC}{CB}=\frac{Q}{P}, \text{ or } P \cdot AC=Q \cdot CB.$$

Therefore, the resultant of two parallel forces is equal to their algebraic sum, is parallel to the forces, and divides the line joining their points of application in the inverse ratio of the magnitudes of the forces.

The line is divided internally when the forces act in the same direction, and externally when the forces act in opposite directions.

39. The Moment of the Resultant of Two Parallel Forces (except Case III).—From the theorem of Art. 35 the moment

of the resultant of concurrent forces is equal to the sum of the moments of the forces about any line whatever. Therefore, referring to Fig. 68,

$$\begin{aligned}\text{Moment of } R &= \text{moment of } R_1 + \text{moment of } R_2 \\ &= \text{moment of } P + \text{moment of } F_1 + \text{moment of } Q + \\ &\quad\ \text{moment of } F_2 \\ &= \text{moment of } P + \text{moment of } Q.\end{aligned}$$

Therefore the moment of the resultant of two parallel forces about any line is equal to the sum of the moments of the two forces about the line.

40. The Resultant of any Number of Parallel Forces.—By applying repeatedly the method of the preceding articles we obtain the resultant and moment of any number of parallel forces. Hence:

(1) The resultant of any number of parallel forces is equal to their algebraic sum and acts parallel to the forces.

(2) The moment of the resultant of any number of parallel forces with respect to any line is equal to the sum of the moments of the parallel forces with respect to the line.

These two facts are sufficient to determine completely the resultant, including magnitude, direction, and line of action.

41. Coordinates of the Center of any Number of Parallel Forces.—In Art. 38 the position of the point C, the point of application of the resultant of two parallel forces, acting at A and B, is independent of the direction of the forces but depends only upon the points of application and the magnitude of the forces. In like manner the point of application of the resultant of any number of parallel forces, if their points of application are given, is independent of the direction of the forces. The point is called the center of the parallel forces. Its coordinates may be found as follows:

Let F_1, F_2, F_3 . . . . be the parallel forces, their points of application (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) . . . . (Fig. 69). Let $(\bar{x}, \bar{y}, \bar{z})$ be the center fo the parallel forces. Since the center is independent of the direction of the forces, we shall assume the forces parallel to OZ. Taking moments about OY, we have

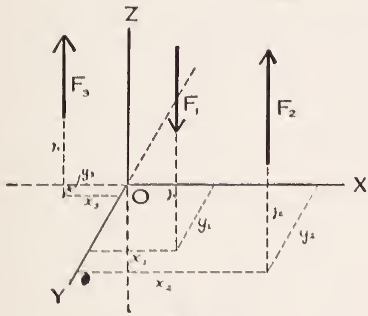


FIG. 69.

$$\bar{x}(F_1 + F_2 + F_3 \ldots\ldots) = F_1x_1 + F_2x_2 + F_3x_3 \ldots\ldots$$

$$\bar{x} = \frac{F_1x_1 + F_2x_2 + F_3x_3 \ldots\ldots}{F_1 + F_2 + F_3 \ldots\ldots} = \frac{\Sigma Fx}{\Sigma F}.$$

Taking moments about OX,

$$\bar{y} = \frac{F_1y_1 + F_2y_2 + F_3y_3 \ldots\ldots}{F_1 + F_2 + F_3 \ldots\ldots} = \frac{\Sigma Fy}{\Sigma F}.$$

By letting the forces act parallel to OX and taking moments about OZ,

$$\bar{z} = \frac{F_1z_1 + F_2z_2 + F_3z_3 \ldots\ldots}{F_1 + F_2 + F_3 \ldots\ldots} = \frac{\Sigma Fz}{\Sigma F}.$$

Problems

98. Forces of 6 and 10 pounds have parallel lines of action 18 inches apart. Find the magnitude of their resultant and the position of its line of action: (a) when they have the same direc-

tion; (b) when they have opposite directions. (In each case give the distance of R from the 10-pound force.)

99. The lines of action of parallel forces, $F=2$, $F=3$, $F=1$, $F=-4$, intersect a perpendicular plane in the following points: $(1, -1)$, $(0, 4)$, $(-1, 3)$, $(1, 2)$. What is their resultant and where does it act?

100. Weights are hung on the rim of a wheel at the mid-point of each quadrant arc as follows: 1st quadrant 5 pounds, 2d quadrant 20 pounds, 3d quadrant 15 pounds, 4th quadrant 10 pounds. What is the smallest additional weight that will keep the wheel from turning?

101. The 16-foot beam AB is supported at P and Q (Fig. 70).

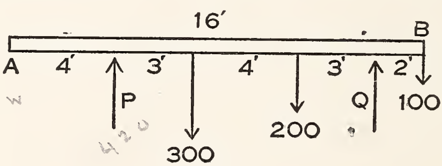


FIG. 70.

It weighs 210 pounds, and also carries the three loads, 200, 300, and 100 pounds, respectively, as per sketch. Find the weight that must be hung at A in order that the supporting force P may be 420 pounds.

102. A table top is an equilateral triangle with legs at A, B, and C (Fig. 71). 40 pounds rests at H (5 feet from C), and 60 pounds at E (4 feet from B). Find the total pressure on each leg. (Neglect weight of table top.)

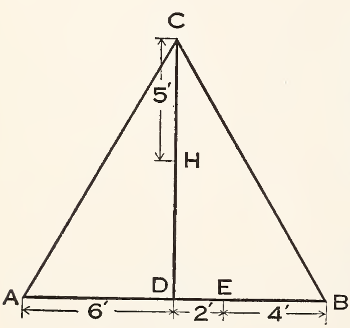


FIG. 71.

103. This same table top (dimensions in feet) weighs 7 pounds per square foot and carries the same loads, but is supported by three legs (each in middle of a side). Find the pressure on each leg.

104. Given a rectangle 6′ x 8′ and forces acting as shown in Fig. 72. Find moment of the forces about O.

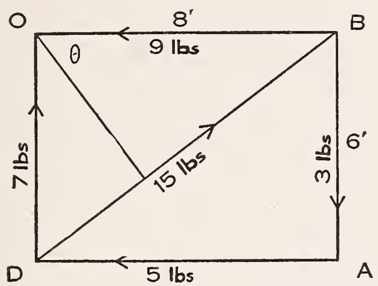


FIG. 72.

42. Case III. Couples.—We shall now consider the effect on a rigid body of two forces which are equal and parallel and act in opposite directions. Two forces, equal in magnitude, parallel, and acting in opposite directions, but not at the same point, are called a couple.

The distance between the lines of action of the forces is the arm of the couple. The plane determined by the lines of action of the two forces is the plane of the couple.

If two equal and parallel forces act in opposite directions on a rigid body, they cannot be replaced by a single force, but their effect is not zero, for they have a tendency to turn the body around. In other words, their moment is not zero.

43. Moment of a Couple.—The sum of the moments of the forces of a couple about any point of the body in the plane of the couple is constant and equal to the product of its arm and one of the forces. In Fig. 73 let P be the couple and O a point in line with the arm of the couple. Taking moments about O of the forces, we have

$$
\begin{aligned}
\text{Moment} &= P \cdot OA - P \cdot OB \\
&= P(OB + BA) - P \cdot OB \\
&= P \cdot BA.
\end{aligned}
$$

It can be shown that the effect on a rigid body is unchanged if one couple is replaced by another having the same moment and acting in the same plane or a parallel plane. The proof of this

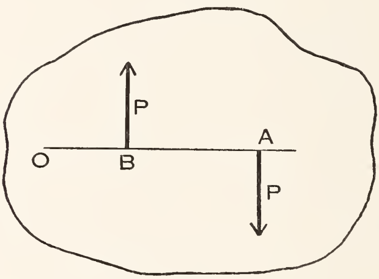


FIG. 73.

is by repeated applications of the principles of Art. 30. It is of an elementary character, but is omitted here on account of its length.

44. Composition of Couples.—*Theorem.*—Any number of couples is equivalent to a single couple. Consider the plane M with the couples P and R (Fig. 74). Substitute for these two couples two other couples having the same arm c such that

$$P \cdot a = S \cdot c$$

and

$$R \cdot b = T \cdot c.$$

Now at A there are acting two forces, the resultant of which is

the algebraic sum of the forces S and T. And at B the same resultant of forces acts. Hence the result of the substitution is a couple. Any number of couples may be treated in the same way.

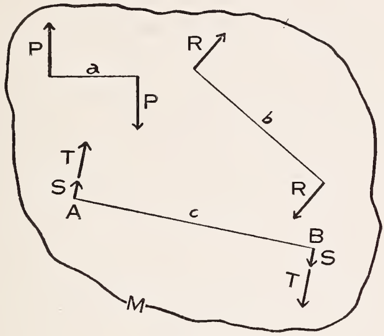


Fig. 74.

45. A couple and a force acting in the same plane or parallel planes may be replaced by a single force whose line of action is determined as follows:

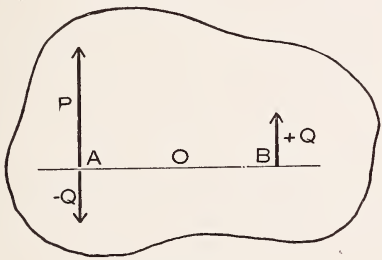


Fig. 75.

By Art. 43 we may replace the couple by a new couple which has one force acting along the line of action of the force P; then the resultant is equal to $(P-Q)+Q=P$, acting parallel to the

original force. Let O be the point at which the resultant force acts. Taking moments around this point, we have

$$(P-Q)\cdot OA=Q\cdot AB.$$

or

$$P\cdot OA=Q(AO+OB)=Q.AB.$$

Therefore OA = moment of couple divided by force P. Hence: The resultant of a couple and a force is a force whose magnitude is that of the original force, acting at a distance from the original force equal to the moment of the couple divided by the force.

46. *Theorem.*—Any number of coplanar forces acting on a rigid body is equivalent to a force acting at any desired point in the plane and a couple. Let M be the point and F any one of the

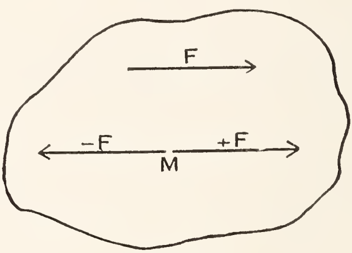


FIG. 76.

forces (Fig. 76). Introduce two forces at M parallel and equal to F but acting in opposite directions. Then the force F has been replaced by a force F, acting at M, equal to the original force F and a couple. The same thing can be done with each of the other forces. The forces at M have a single resultant force acting at M, and the resultant of the couples is a single couple. (Art. 44.)

It can also be shown that any non-coplanar system of forces acting on a rigid body is equivalent to a single force acting at any desired point and a couple.

Problems

105. Given the following couples, convert to equivalent couples by finding the forces corresponding to the indicated arms.

Force	3 lbs.	6 lbs.	12 lbs.	14 lbs.
Arm	8 ft.	7 ft.	12 ft.	6 ft.
New arm	6 ft.	21 ft.	7 ft.	20 ft.

106. Convert the following couples to equivalent couples by finding the arms corresponding to the indicated forces.

Force	5 lbs.	6 lbs.	10 lbs.	12 lbs.
Arm	7 ft.	9 ft.	9 ft.	17 ft.
New force	8 lbs.	9 lbs.	4 lbs.	5 lbs.

107. Given two couples in the same plane, both having the same direction, namely 5-pound forces at 6 feet, and 7-pound forces at 8 feet. Find the force of the resulting couple, if the arm is taken as 5 feet.

108. Given these two couples in parallel planes and of opposite directions: 3 pounds at 9 feet, and 18 pounds at 6 feet, find the moment of the resulting couple. Also find the arm if the force is taken as 9 pounds.

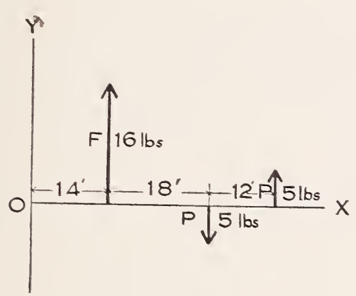


FIG. 77.

109. Given the force F and the couple P and P acting as in Fig. 77. Find the magnitude, position, and direction of the single force which is the equivalent of the system.

Review Problems

110. A 60-pound cylindrical cask is 4 feet high and 3 feet in diameter. How long an iron rod weighing 4 pounds per foot will just overturn the cask, placed as per sketch (Fig. 78)?

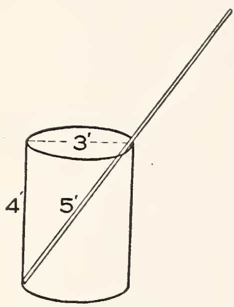


FIG. 78.

111. An iron rod weighs 10 ounces per inch length. It is lying on a table with $3\frac{1}{4}$ feet projecting over the edge, and a 50-pound weight presses on the end lying on the table. How long is the rod if it is just about to turn?

112. A rod 8 feet long and weighing 40 pounds is supported at each end. How far from one end must a weight of 70 pounds be hung in order that the supporting force at that end be 50 pounds?

113. The sides of a triangular table are 3, 4, and 5 feet long. A weight of 24 pounds is placed on the table 8 inches from the 3-foot side and 18 inches from the 4-foot side. What is the thrust in each leg? (The legs are exactly at the corners. Neglect weight of table.)

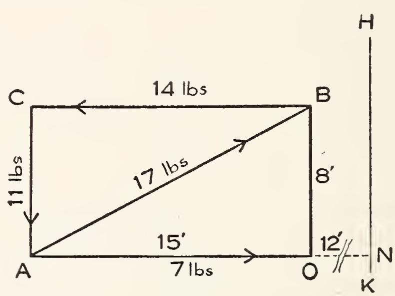


FIG. 79.

114. Given forces acting as in Fig. 79, also, $AO = 15$ feet, $OB = 8$ feet, and $ON = 12$ feet. Find the single force, acting in the line HK which will have the same turning moment about the point O as the four given forces acting together.

115. Given the force F, and the couple Q and Q, acting as in Fig. 80. Find the magnitude, position, and direction of the single force which is equivalent to this system.

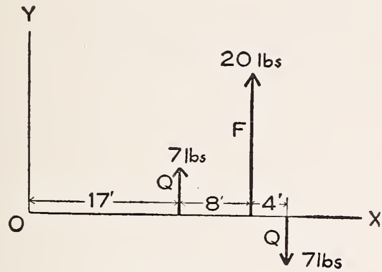


FIG. 80.

116. A brakeman sets a brake by pulling 50 pounds with one hand and pushing 50 pounds with the other. His forces act tangentially to the brake wheel, whose diameter is 18″. Another time he gets the same result by using a lever in the handwheel and pulling 25 pounds. How far from the rim of the handwheel must his hands be placed?

117. Parallel forces, each equal to P, act in the same direction at three corners of a square, perpendicular to its plane. At the other corner such a force acts that the whole system is a couple. Determine the moment of the couple if the side of the square is a.

CHAPTER IV

STATICS OF A RIGID BODY

47. Equilibrium of a Rigid Body.—The resultant of a number of forces in a plane is either a single force or a couple. (Arts. 45, 46.) Therefore, the condition for equilibrium is that the single force shall be zero and that there shall be no unbalanced couple. The component of the resultant force in any direction equals zero if the sum of the components of the several forces in that direction equals zero. That the resultant force may equal zero, its component parts in two different directions must equal zero. In this case there can be no resultant except a couple, and this will vanish when its moment about any point is zero. This can be expressed by the following equations:

$$\Sigma X = 0.$$
$$\Sigma Y = 0.$$
$$\Sigma M = 0.$$

Thus a necessary and sufficient condition for equilibrium of a number of forces in a plane is:

If the sums of the resolved forces in any two directions each equals zero and if the sum of the moments about any point is also zero, the system of forces is in equilibrium.

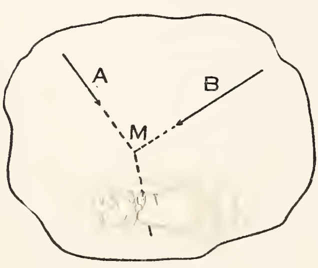


Fig. 81.

48. *Theorem.*—If a rigid body is in equilibrium under three forces, these forces pass through a point or are parallel.

Let A and B intersect at M

(Fig. 81). The moments of A and B about M equal zero. Since the body is in equilibrium, the sum of the moments is zero. Therefore the moment of C about M is zero. Hence C must pass through M.

If the two forces A and B are parallel, their resultant will be parallel to them; and if the body is in equilibrium, the third force C must be equal and opposite to this resultant and have the same line of action.

49. The foregoing principles are illustrated by the following examples:

Example 1.—A uniform rod 12 feet long is supported by a smooth horizontal peg and the pressure of a smooth vertical wall against one end of the rod. The peg is 3 feet from the wall and the rod is in equilibrium. Find the angle θ which the rod makes with the horizontal.

Since the wall and the peg, P (Fig. 82), are smooth, the reaction, R, of the wall against the rod acts at right angles to the wall, and the reaction, S, of the peg against the rod acts at right angles to the rod. The weight, W, of the rod may be regarded as acting at the center of the rod. Taking moments about A,

$$S \cdot 3 \sec \theta = W \cdot 6 \cos \theta.$$

Resolving vertically,

$$S \cos \theta = W,$$
$$\therefore \cos^3 \theta = \tfrac{1}{2},$$
$$\theta = 37°28'.$$

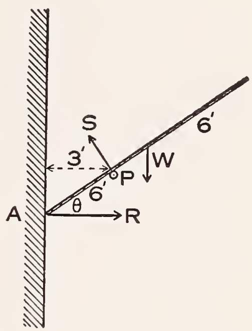


FIG. 82.

Example 2.—A trap door 3 feet square weighing 50 pounds is held partly open by a cord 4 feet long. The cord is perpendicular to the plane of the door in this position and its ends are made fast to the middle of the door's outer edge and to a hook in a vertical plane through the hinges. Find the tension in the cord and the magnitude and direction of the thrust on the hinges.

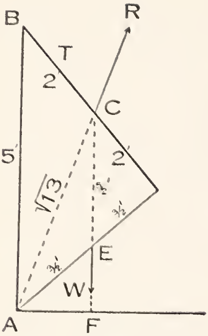


Fig. 83.

Since the door is in equilibrium under three forces, the reaction, R, passes through C, the intersection of T and W (Fig. 83). ABC is the force triangle.

$$T : W = 2 : 5, \qquad \therefore T = \frac{2W}{5} = 20 \text{ lbs.}$$

$$R : W = \sqrt{13} : 5, \quad \therefore R = \frac{W\sqrt{13}}{5}$$

$$= 10\sqrt{13} = 36.1 \text{ lbs.}$$

$$FC = FE + EC = 0.9 + 2.5 = 3.4,$$

$$\tan\theta = \frac{FC}{AF} = \frac{3.4}{1.2} = 2.83, \qquad \therefore \theta = 70^\circ 34'.$$

Problems

118. A man weighing 180 pounds is half-way up a 20-foot ladder weighing 80 pounds and resting against a smooth wall. The foot of the ladder rests upon a smooth floor, and is prevented from slipping by a horizontal cord attached to the wall. Find the tension of the cord, if the angle between the ladder and the floor is 30°.

119. A uniform beam AB is 6 feet long, weighs 25 pounds, and rests at A upon a smooth floor, and at D upon the smooth top of the vertical post CD, 3 feet high. A weight of 50 pounds hangs from B, and the end A is prevented from slipping by a cord AC, 4 feet long, connecting it with the foot of the post. Find the tension of the cord and the reaction of the floor at A.

120. A uniform rod 22 feet long, weighing 80 pounds, rests with its upper end A against a smooth vertical wall, and its lower end supported by a cord, 26 feet long, attached to a point directly over A. Find the tension of the cord, and the resistance of the wall.

121. A trap door 3 feet square is held at an inclination of 30° to (and above) the horizontal plane through its hinges by a cord attached to the middle of the side opposite the hinges. The other end of the cord, which is 5 feet long, is attached to a hook vertically above the middle point of the hinged side of the door. Find

the tension in the cord, and the direction and magnitude of the pressure between the door and its hinges, the weight of the door being 50 pounds, which may be taken as acting at the center of the door.

122. The supporting cross bar, AB, of a platform is 6 feet long and holds weights as shown (Fig. 84). The inner end is fastened to the wall by a hinge at A. A cable BC at an angle of 30° with the vertical supports the other end. Solve for the tension in the cable BC and the reaction at A in amount and direction.

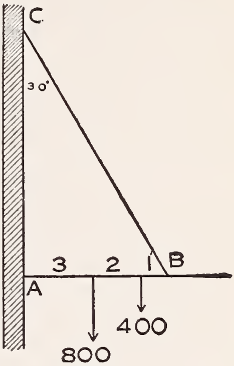


FIG. 84.

123. A uniform rod 10 feet long weighing 20 pounds rests with one end at the junction of a vertical wall and a horizontal floor; from a point 2.5 feet from the other end a cord runs horizontally to a point 6 feet above the floor. Find the tension of the cord and the amount and direction of the reaction at the lower end of the rod.

50. Two-Force Pieces.—Many structures, such as roof and bridge trusses, are made up of members connected by pins, and, neglecting the weight of the members themselves, each member is acted upon by only two forces, acting at the joints. Since each member is in equilibrium, these two forces must be equal and opposite in direction, that is, the line of action of the forces is the axis of the member. These are called two-force pieces. If a member is in tension it is called a tie, and if it is in compression it is called a strut.

51. Method of Sections.—The method of sections is a method used to determine the internal stresses in the members of the structure. It consists in cutting a section through a structure and considering the part on one side of the section as a rigid body, held in equilibrium by the external forces acting on this side of

6

the section and the stresses along the members cut by the section. To explain more fully, we will find some stresses by this method.

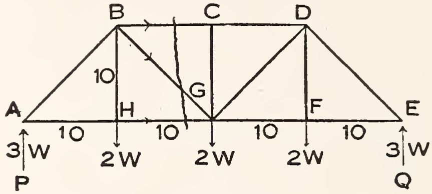


FIG. 85.

Consider the forces acting as illustrated by the figure (Fig. 85). First find the reactions at A and E. Taking moments of all the external forces about A, we have

$$Q \cdot 40 = 2W(10+20+30),$$
$$Q = 3W.$$

Therefore

$$P = 3W.$$

Let it be required to find the stress in BG. In Fig. 86 let

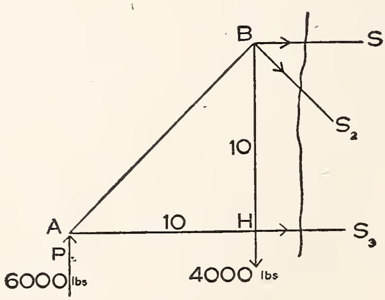


FIG. 86.

$W = 2000$ pounds. Take a section cutting BC, BG, and HG, as shown. Indicate the forces as acting away from the section.

To get S_2 resolve vertically $(\Sigma Y = 0)$.

$$6000 - 4000 - S_2 \cos 45° = 0.$$

Therefore

$$S_2 = 2000\sqrt{2} \text{ lbs.}$$

Since the result is positive, the stress acts away from the joint B, and therefore is tension. If the sign had been negative, the stress would have been toward B and therefore compression.

We can find S_3 by taking moments about B, the intersection of S_1 and S_2.

$$6000 \cdot 10 - S_3 \cdot 10 = 0;$$
$$S_3 = 6000 \text{ lbs.}$$

The positive sign indicates tension as before. To find S_1, resolve horizontally $(\Sigma X = 0)$.

$$S_3 + S_2 \cos 45° + S_1 = 0;$$
$$6000 + 2000 + S_1 = 0;$$
$$S_1 = -8000 \text{ lbs.}$$

The negative sign indicates that the stress acts toward B and is therefore compression.

If we had wished to compute only one force, say S_1, the simplest method would usually be by taking moments about the intersection of the other two unknown stresses.

Thus:

$$S_1 \cdot 10 + 6000 \cdot 20 - 4000 \cdot 10 = 0;$$
$$S_1 = -8000;$$
$$= 8000 \text{ lbs. compression, as above.}$$

Problems

124. In the truss of sketch (Fig. 87) the peak carries a concentrated load of 800 pounds, and in addition to that the left rafter carries 60 pounds per foot run. Find the stress in the loaded rafter by means of a section.

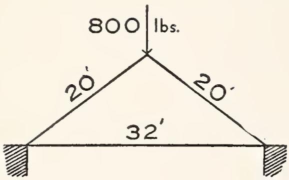


FIG. 87.

125. The truss of sketch (Fig. 88) carries 200 pounds per foot run over *PAQ*. Find the amount and kind of stress in the vertical member by means of a section.

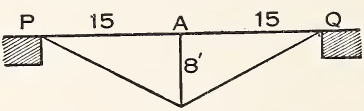


FIG. 88.

126. The truss of sketch (Fig. 89) carries two concentrated loads, as shown. Find by the method of sections the amount and kind of stress in the member marked (*).

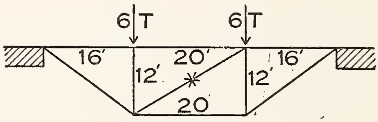


FIG. 89.

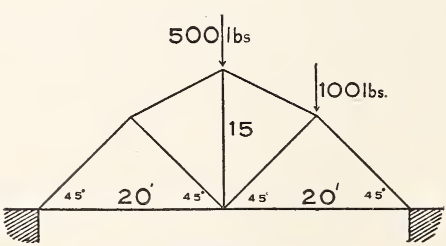


FIG. 90.

127. In the truss of sketch (Fig. 90) there are two concentrated loads carried, as shown. Find by the method of sections the amount and kind of stress in the member between the loads.

128. The girder of the sketch (Fig. 91) carries three concentrated loads, as shown. Find the amount and kind of stress in the two members marked (*), using the method of sections.

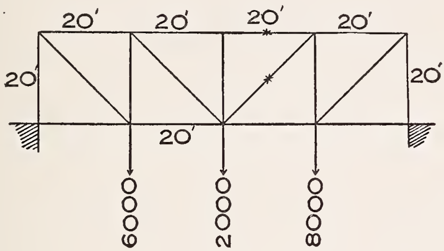


FIG. 91.

129. Given the cantilever conveyor of the sketch (Fig. 92). Find the amount and kind of stress in the two members marked (*) when the frame is loaded as shown.

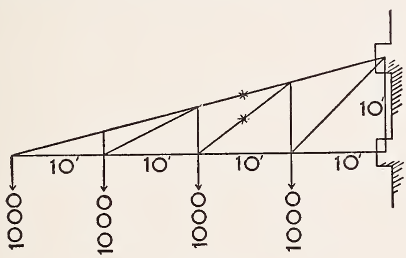


FIG. 92.

52. Maxwell's Method.—The graphic method, invented by Clerk Maxwell, is based upon the fact that since the forces are in equilibrium they may be represented in amount and direction by the consecutive sides of a closed polygon. If the loads and

external reactions are all vertical, the polygon of the external forces reduces to a vertical straight line.

Let the loads be as indicated in the frame diagram (Fig. 93).

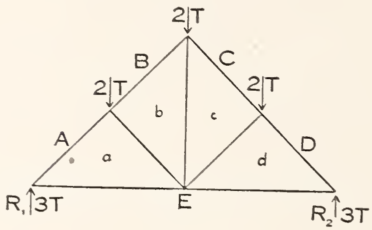
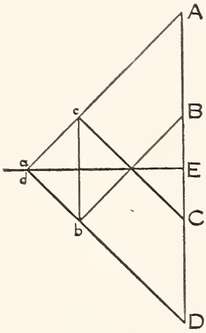


FIG. 93.



$Aa = Dd = 3\sqrt{2}$ (C)
$Bb = Cc = 2\sqrt{2}$ (C)
$ab = cd = \sqrt{2}$ (C)
$Ea = Ed = 3$ (T)
$bc = 2$ (T)

FIG. 94.

To name these forces, place a capital letter between successive external loads and reactions and a small letter in each space made by the members of the frame. In this way each load and each member is straddled by a pair of letters. Each force is designated by the letters on each side of it.

To draw the reciprocal diagram (R. D.), first lay off to scale the external forces and reactions in order. Thus the loads between A and B, B and C, C and D, and the reactions between D and E, and E and A, in the frame diagram, are represented in amount and

direction by the lines AB, BC, CD, DE, and EA, in the reciprocal diagram (Fig. 94), forming a closed line. To represent the internal stresses in the members of the frame on the R. D., we draw lines parallel to the members of the frame. All the forces acting at a joint in the frame form in the R. D. a closed polygon. Consider the forces acting at the right support R_2. We have $DE = R_2$, already drawn. At E in the R. D. draw a line Ed parallel to the horizontal member Ed. Then d will lie somewhere on the line Ed. From D in the R. D. draw Dd parallel to Dd of the frame. d will lie at the intersection of these two lines. The triangle DEd is thus the force polygon for the three forces acting at R_2. Next draw the force polygon for the joint R_1. Draw Ea of the R. D. parallel to Ea of the frame and from A draw Aa parallel to Aa of the frame. a will lie at their intersection and will coincide with d. To determine c draw from C a line Cc parallel to Cc of the frame, and from d draw dc parallel to dc of the frame. Their intersection determines c. In like manner the intersection of Bb and ab determines b. The line joining b and c will be vertical and parallel to bc of the frame, and checks the accuracy of the drawing.

To determine the *kind* of stress in a member of the frame, let us consider again the forces acting at R_2. These forces will form a closed polygon, viz., the triangle DEd in the R. D. (Fig. 95). The sides of the triangle represent in amount and direction the forces acting at R. Since we know DE acts upward, Ed acts to the left, and dD acts downward, as indicated by the arrows. Since Ed acts to the left, it acts away from the joint, and therefore Ed is in tension. Since dD acts downward, the stress is toward the joint, and therefore dD is in compression.

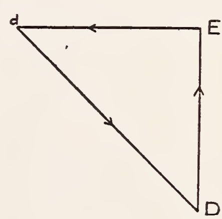


FIG. 95.

Consider the stresses at the joint between C and D. These are represented by the polygon $CDdc$ (Fig. 96) in the R. D. We know that CD acts downward. This gives the forces and directions as indicated by the arrows. It should be noted that the arrow on Dd here points upward, while in Fig. 95 it points downward. But here we are considering the forces at the joint between C and D, and the force is indicated as acting toward the joint, showing compression. In the former figure we were considering the forces at the point of support; the stress in Dd acted toward the support, and indicates compression in both cases.

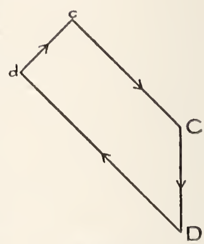


FIG. 96.

Problems

130. The rafters of a simple triangular roof truss are 20 feet long, and the span is 32 feet (Fig. 97). Find the amount and

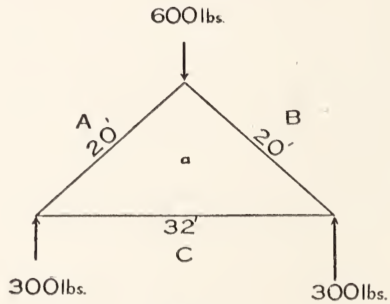


FIG. 97.

kind of stress in each member when loaded with 600 pounds at the peak. Use the R. D., and find the stress in the left-hand rafter (as a check) by the method of sections.

131. The roof truss of sketch (Fig. 98) has a vertical member, and carries 2000 pounds at the peak. Find amount and kind of stress in the two members marked (*).

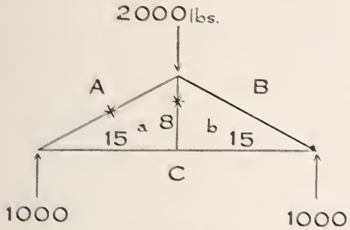


FIG. 98.

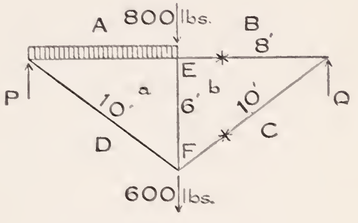


FIG. 99.

132. Find the amount and kind of stress in the members marked (*) in the five-member truss of sketch (Fig. 99). The joints E and F carry concentrated loads of 800 pounds and 600 pounds respectively, and the member PE carries a uniform load of 100 pounds per foot run.

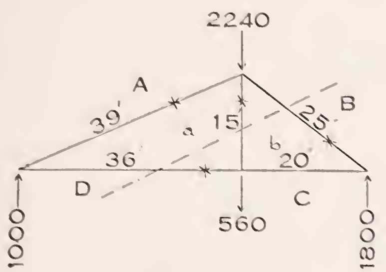


FIG. 100.

133. The sketch shows a roof truss with a vertical member (Fig. 100). The load on the rafters is 70 pounds per foot, and the load on the horizontal members (due to flooring) is 20 pounds per foot. Find the amount and kind of stress in the four members marked (*), and check the stress in the vertical member by taking a section.

134. This roof truss (Fig. 101) carries 21 tons at the peak,

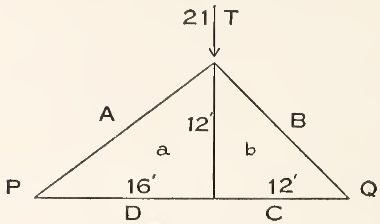


FIG. 101.

and a uniform load of ½ ton per foot run over PQ. Find amount and kind of stress in all five members.

135. A truss with 8-foot vertical member is shown in sketch (Fig. 102). The 17′ rafter carries 200 pounds per foot, and the

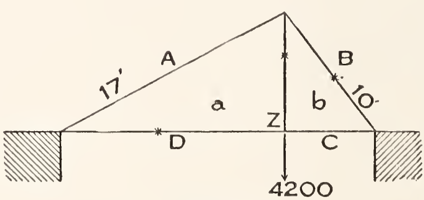


FIG. 102.

10′ rafter 80 pounds per foot. Also 4200 pounds acts at joint Z. Find amount and kind of stress in members marked (*).

53. Three-Force Pieces.—Heretofore we have considered cases where members of a structure are acted upon by two forces only and the members have been subject to either compression or tension. If a member is acted upon by three or more forces, usually none of them will act along the axis of the member. Consider the following figure to illustrate the problem:

Find the tension in the rope BC and the amount and direction of the reaction at A, if $W = 2800$ pounds (Fig. 103). The rod

AB is acted upon by three forces: The tension T in BC, acting in direction BC; the force W, acting downward at D, and re-

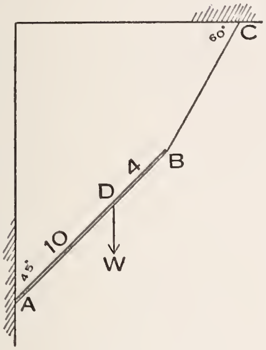


FIG. 103.

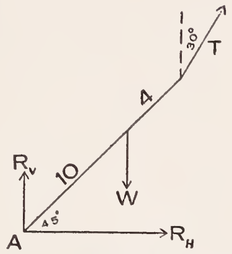


FIG. 104.

action at A, unknown in amount and direction. We will resolve this reaction into its horizontal and vertical components, R_H and R_V, as shown in the Fig. 104.

To find the tension T, take moments about A.

$$T \cdot 14 \sin 15° = 2800 \cdot 10 \cos 45°;$$
$$T = 5464.1 \text{ lbs.}$$

Resolving horizontally, $(\Sigma X = 0)$,

$$T \cos 60° + R_H = 0 \qquad \therefore R_H = -2732.1 \text{ lbs.}$$

Resolving vertically, $(\Sigma Y = 0)$,

$$T \cos 30° - 2800 + R_V = 0 \qquad \therefore R_V = -1932.1 \text{ lbs.}$$

The negative signs of R_H and R_V indicate that the actual directions of these forces are opposite from what is assumed in the figure.

$$\theta = \tan^{-1} \frac{-1932.1}{-2732.1} = 35°16' + 180° = 215°16'.$$
$$R = R_H \sec \theta = 3346.2 \text{ lbs.}$$

Problems

136. Find the stresses and reactions in the frame of sketch, Fig. 105, loaded as shown. Neglect the weights of the members.

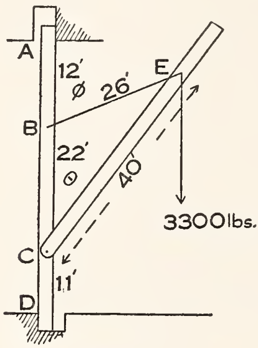


FIG. 105.

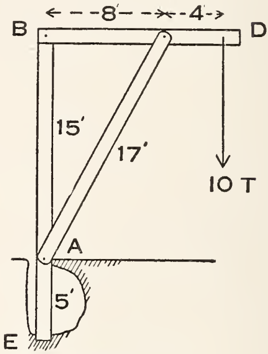


FIG. 106.

137. Find the stresses and reactions in the framework of the sketch, loaded as shown (Fig. 106). Neglect weights of the members.

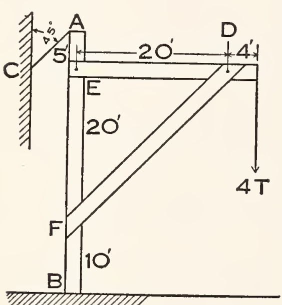


FIG. 107.

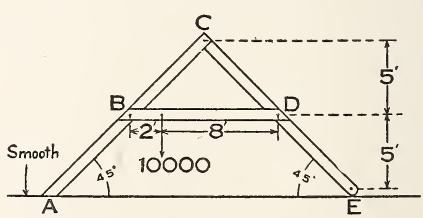


FIG. 108.

138. In the crane shown in Fig. 107, find the forces acting on the pins and the tension in the tie AC.

139. Find the amount and direction of the pressure of the pin at B on the member ABC, of the pin C on the member ABC, and of the pin at D on the member BD (Fig. 108).

140. A uniform spanker boom weighing 400 pounds and 32 feet long is topped up to 30° with the horizontal by the topping lift which is fast to the boom's end, and in this position makes an angle of 60° with the boom (Fig. 109). Find the tension of the lift, and the amount and direction of the thrust at the gooseneck.

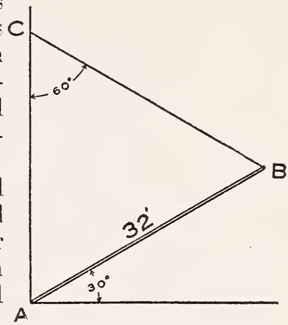


FIG. 109.

141. The upper end of a 150-pound beam rests against a smooth vertical wall, 3 feet above the floor. The lower end is fastened to a smooth hinge in the floor 8 feet from the wall. Find the wall resistance, the hinge resistance, and the angle the latter makes with the floor.

Review Problems

142. In the truss of sketch (Fig. 110) the rafters carry 30 pounds per foot run and meet at right angles. Find the amount and kind of stress in the member marked (*) by means of a section.

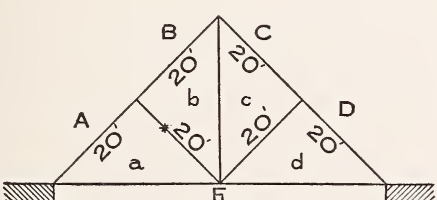


FIG. 110.

143. Find reactions R_1 and R_2, and the stresses in the different members of the truss of sketch (Fig. 111).

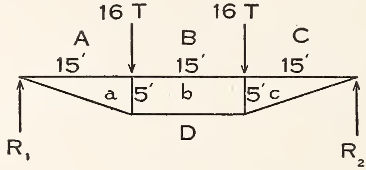


FIG. 111.

144. Points of application of forces are equidistant, and K is on the perpendicular bisector of AD (Fig. 112). Find (1) the magnitude and direction of reaction at G, the joint A being

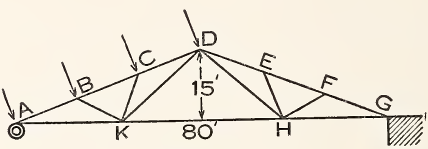


FIG. 112.

mounted on frictionless rollers; (2) the magnitude and direction of reaction at A.

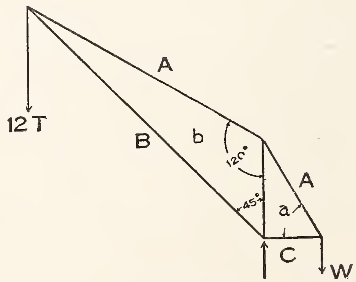


FIG. 113.

145. Determine stresses Bb, Ab, ab, Aa, Ca, and determine W if the stay Aa makes an angle 30° with the vertical. Determine the reaction at BC (Fig. 113).

146. The truss sketched (Fig. 114) carries 40 pounds per foot on the rafters, and 20 pounds per foot on the platform PQ

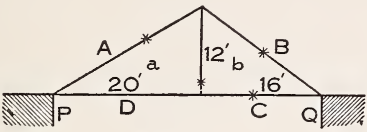


FIG. 114.

(due to flooring). Find the amount and kind of stress in the four members marked (*), using the R. D. Determine the kind of stress in the rafters and the vertical member by drawing a separate sketch of the polygon of forces for the peak joint, as it appears in the R. D. Also, find the stress in the right-hand horizontal member by the method of sections.

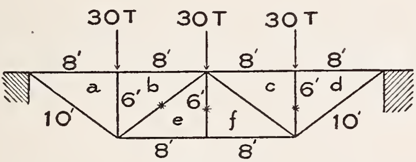


FIG. 115.

147. Draw the R. D. for the truss sketched (Fig. 115), and find amount and kind of stress in the members marked (*). Then find the stress in the left-hand lower horizontal member by means of a section.

148. In the Warren girder of sketch (Fig. 116), PQ carries a load of 50 pounds per foot and also has a concentrated load of 1200 pounds at its center (mid point of PQ). Each member is 10 feet long. Draw the R. D. and find amount and kind of stress in the members marked (*).

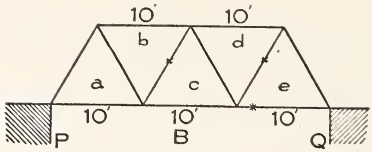


FIG. 116.

149. The Queen Truss of sketch (Fig. 117) carries two loads (9 and 18 tons), as shown. Find amount and kind of stress in the members marked (*).

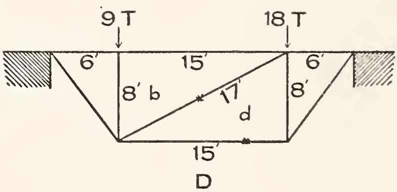


FIG. 117.

150. Given the Queen truss of sketch, loaded over the struts as shown (Fig. 118). Draw the R. D. and compute the stresses

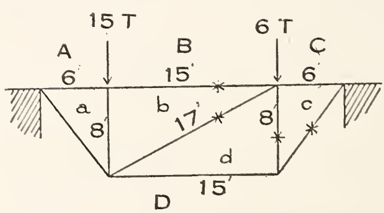


FIG. 118.

in the four members marked (*), and state the kind of stress in those members.

151. The truss of sketch (Fig. 119) carries a concentrated load of 2200 pounds at the peak, also the 15-foot member carries

a load of 80 pounds per foot. Draw the R. D., and find amount and kind of stress in the members marked (*).

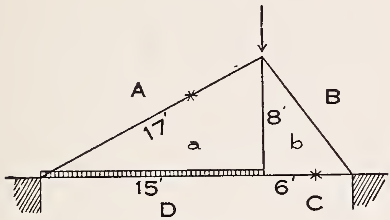


FIG. 119.

152. The A-frame shown in Fig. 120 supports a load of 10,000 pounds at the middle of a member *BD*, which is 18 inches from

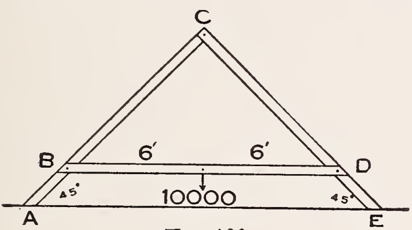


FIG. 120.

the floor. Determine the pin reactions at *B*, *C*, and D caused by this load, if the floor is considered to be smooth.

153. Consider each member of the A-frame in Ex. 152 to weigh 100 pounds per linear foot and determine the pin reactions at *B*, *C*, and *D*.

154. A pentagon *ABCDE*, formed of equal uniform heavy rods, connected by smooth joints at their ends, is supported symmetrically in a vertical plane with *A* uppermost, and *AB*, *AE*, in contact with two smooth pegs in the same horizontal line. Prove that if the pentagon is regular the pegs must divide *AB* and *AE* each in the ratio

$$1+\sin\tfrac{1}{10}\pi : 3\sin\tfrac{1}{10}\pi.$$

CHAPTER V

CENTER OF GRAVITY AND MOMENT OF INERTIA

54. Gravitational Forces.—The formulæ of Art. 41 determine a point on the line of action of the resultant of a system of parallel forces. The forces due to gravity, the pull which the earth exerts on bodies situated on its surface, we may regard without appreciable error as constituting such a system of parallel forces. since the dimensions of any body we shall study are very small when compared with the radius of the earth. Such a system of parallel gravitational forces acting on a body will have a resultant whose line of action, no matter what the position of the body in space, will always pass through the same point in the body; for it was shown (Art. 41) that the point of application of the resultant of a system of parallel forces does not depend on the angle which the lines of action of the forces make with the coordinate planes, but that it is completely determined by the magnitudes of the forces and by their points of application. The moment of the resultant force about this point through which its line of action passes is obviously zero. This leads us to the following definition.

55. Center of Gravity.—The *center of gravity* of a body is a point G, such that if the body be supported by any axis whatever through G, it will have no tendency to turn about that axis. That is, about any axis through G the sum of the moments of the gravitational forces acting on the particles of which the body is constituted must be zero.

56. Coordinates of the Center of Gravity.—In order to determine in a given body this point G, we choose, to suit our con-

venience, three mutually perpendicular planes, constituting the planes of reference for a system of Cartesian coordinates, and regard them as rigidly attached to the body. As the body is moved about in space, these planes move with it and the values of the coordinates x, y, z, which indicate the distances of any point in the body from these planes, are in no way affected by its change in position. In order to determine the three coordinates $(\bar{x}, \bar{y}, \bar{z})$ of the point G we take moments about each of the three axes through G, parallel respectively to the X-, Y-, and Z-axes of

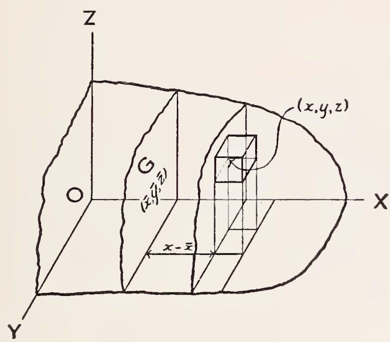


Fig. 121.

the system of coordinates which we have just set up. Place the body with the X- and Y-axes horizontal and the Z-axis, therefore, parallel to the lines of action of the gravitational forces, and consider the tendency of these forces to produce rotation about an axis through G, parallel to the Y-axis (see Fig. 121). We first study the moment due to a piece of the body of mass Δm, of any convenient size and shape, subject only to the restrictions that Δm shall lie between two planes, each parallel to the YZ-plane and distant respectively x and $x + \Delta x$ from it, and that Δm shall approach zero as Δx approaches zero. Denote the resultant

of the gravitational forces acting on Δm by ΔW. Then $\Delta W = k\Delta m$, where k is a constant for all bodies at the same place on the earth's surface. (The value of k also depends upon the units of force and mass employed, see Art. 122.) The distance of the line of action of ΔW from the Y-axis is between x and $x+\Delta x$. Therefore, taking moments about an axis through G parallel to OY, the arm is $x+\epsilon\cdot\Delta x-\bar{x}$ and

$$\text{Moment of } \Delta W = (x-\bar{x})\cdot k\cdot\Delta m+\epsilon k\cdot\Delta x\cdot\Delta m,$$

where ϵ is less than 1. The sum of the moments of all the gravitational forces acting on the body is given by

$$\underset{\Delta x=0}{\text{Limit}}\sum\left[(x-\bar{x})\cdot k\cdot\Delta m+\epsilon k\cdot\Delta m\cdot\Delta x\right],$$

where the summation is to cover the entire mass. By theorems of the Calculus this sum is equal to the definite integral of $(x-\bar{x})\cdot k\cdot dm$, taken between the proper limits so that the integration is extended over the entire mass of the body. Whether this calls for single, double, or triple integration, we shall denote it by

$$\int_M (x-\bar{x})\cdot k\cdot dm.$$

From the definition of the center of gravity (Art. 55) the sum of the moments represented by this integral is zero. Hence

$$(1)\qquad \int_M (x-\bar{x})\cdot k\cdot dm=0.$$

Proceeding in a similar manner, taking moments about an axis through G parallel to the X-axis, we obtain

$$(2)\qquad \int_M (y-\bar{y})\cdot k\cdot dm=0.$$

Placing the body so that the X- and Z-axes are horizontal and proceeding as before, we get

$$(3) \qquad \int_M (z-\bar{z}) \cdot k \cdot dm = 0.$$

Since $\bar{x}$, $\bar{y}$, and $\bar{z}$, and k are constants for the integration, we can solve the equations (1), (2), (3) and obtain

$$(4) \qquad \bar{x} = \frac{\int_M x\,dm}{\int_M dm}, \quad \bar{y} = \frac{\int_M y\,dm}{\int_M dm}, \quad \bar{z} = \frac{\int_M z\,dm}{\int_M dm}.$$

The denominator $\int_M dm$ is the mass M of the body.

57. In future applications of the summation theorem of the Calculus we shall not indicate the several steps of the procedure as in the last paragraph, but describe it in some such manner as this: Take an element of mass dm at a distance x from the YZ-plane. The moment of the gravitational force acting on it is $k \cdot x \cdot dm$. Integrating to cover the entire mass, we have $\int_M k \cdot x \cdot dm$, or the sum of the moments of all the gravitational forces acting on the body.

The derivation of equations (4) shows that the size and shape of dm are not fixed. We choose an infinitesimal element of length area, or volume—ds, dA, or dV respectively—as the problem in hand may require, precisely in the way already familiar in the corresponding problems of finding length of arc, area, or volume by methods of the Calculus. We are subject only to the additional restriction that in computing $\bar{x}$, $\bar{y}$, $\bar{z}$, all points of this infinitesimal element shall in the limit be equally distant respectively from the YZ, XZ, or XY planes. Then dm is

equal to ds, dA, or dV, as the case may be, multiplied by the density factor ρ, where ρ is the mass of a unit volume.

58. If we write the first of equations (4) as $\int_M x \cdot k \cdot dm = \bar{x} \cdot k \cdot M$, the left member gives the sum of the moments about the Y-axis of all the gravitational forces acting on the body of mass M; the right member is the moment of the resultant of all these forces acting at the point G. Since we may take any line as the Y-axis, this shows that about any axis the moment of the gravitational forces acting on the body can be replaced by the moment of the force due to a particle endowed with a mass equal to that of the given body and situated at its center of gravity.

59. Center of Mass; Centroid.—It is apparent from equations (4) that the point G is determined by the distribution of the mass of a body quite independently of the action of the gravitational forces, and that, consequently, the point G might with equal correctness be called the *center of mass* of the body. Further, if the mass be uniformly distributed through the space occupied by the body, that is, if ρ be constant, the density factor ρ can be cancelled from the numerator and denominator in equations (4), thus giving $\bar{x}$, $\bar{y}$, $\bar{z}$ a purely geometric definition. The center of gravity of a purely geometric quantity, arc, area or volume, is sometimes called its *centroid.* Any plane, axis, or point, of symmetry contains the centroid, a fact which should be made use of whenever possible to avoid unnecessary computation.

60. *Illustrative Examples.*

Some of the further simplifications that can be made use of in applying these formulæ will be shown in the examples that follow. In particular, whenever the value of the denominator is a well-known result from the Geometry or the Calculus, only the numerator will need actually to be integrated.

Example 1.—Find the center of gravity of a plate weighing w pounds per square foot, bounded by the parabola $y^2=8x$ and the double ordinate for which $x=2$ feet.

Solution.—The graph of one face of the plate is shown in Fig. 122. If we consider the thickness of the plate it is evident from considerations of symmetry that the center of gravity will lie in the plane equidistant from the faces of the plate, but it is customary in the case of thin plates to neglect the thickness and proceed as if we had to consider an area only. Then $\bar{z}=0$, and since the X-axis is an axis of symmetry for the area we have also $\bar{y}=0$. To find $\bar{x}$ we take as the element of area a strip dx wide parallel to the Y-axis, whence $dA=2y\,dx$. Since $\rho=w$, we have

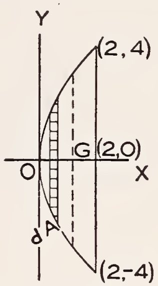


FIG. 122.

$$dm=\rho dA=2yw\ dx.$$

Substituting in the first of formulæ (4),

$$\bar{x}=\frac{\int_M x\,dm}{\int_M dm}=\frac{\int_0^2 2wxy\,dx}{\int_0^2 2wy\,dx}=\frac{2\sqrt{2}\int_0^2 x^{\frac{3}{2}}\,dx}{2\sqrt{2}\int_0^2 x^{\frac{1}{2}}\,dx}=\tfrac{6}{5}.$$

Therefore the center of gravity is on the axis of the plate, $\frac{6}{5}$ feet from its vertex.

Example 2.—Find the center of gravity of a rod l feet long, whose density varies as the distance from one end, the weight increasing from 0 pounds per foot run to w pounds per foot run.

Solution.—Since the only variable dimension that enters into

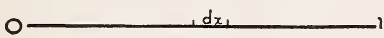


FIG. 123.

this problem is the distance from one end of the rod, we may regard it as a straight line having mass. Then

$$\rho=x/l\cdot w,\ dm=x/l\cdot w\cdot dx,$$

whence, cancelling a constant factor common to numerator and denominator.

$$\bar{x} = \frac{\int x\,dm}{\int dm} = \frac{\int_0^l x^2\,dx}{\int_0^l x\,dx} = 2l/3.$$

Example 3.—Find the center of gravity of a right circular cone when the density of the cone varies as the distance from a plane through the vertex, and parallel to the base.

Solution.—Suppose the cone to have altitude h and base radius a. Take the axis of the cone as the X-axis, the origin at the ver-

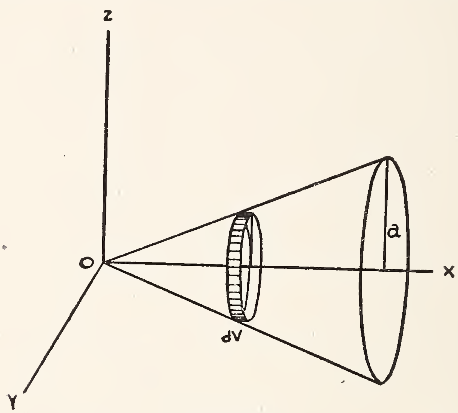


FIG. 124.

tex of the cone. From symmetry $\bar{y}=0$, $\bar{z}=0$. To find $\bar{x}$, take as the element of volume dV, a cylindrical slice dx thick, with bases parallel to the YZ-plane.

Then

$$\rho = kx,\ dV = \pi z^2 dx,$$

whence

$$dm = \rho dV = k\pi x z^2 dx.$$

From Fig. 124 $z/x = a/h$. Then

$$\bar{x}=\frac{\int x\,dm}{\int dm}=\frac{k\pi\int_0^h x^2z^2dx}{k\pi\int_0^h xz^2dx}=\frac{\int_0^h x^4dx}{\int_0^h x^3dx}=4h/5.$$

The student should note that the *method* of this example is applicable to finding the center of gravity of any solid of revolution.

Example 4.—Find the center of gravity of the area bounded by the cardioid $r=a(1+\cos\phi)$, and the initial line OX.

Solution.—Take an element of area with dimensions dr by $r\cdot d\phi$ and integrate first with respect to r to include the sector OAB (Fig. 125), and then with respect to ϕ from 0 to π to sum all such sectors.

$$dA=r\cdot d\phi\cdot dr,$$

and since

$x=r\cos\phi,\ y=r\sin\phi,$

we have

$$\bar{x}=\frac{\int x\,dm}{\int dm}=\frac{\int_0^\pi\int_0^{a(1+\cos\phi)} r^2\cos\phi\,dr\,d\phi}{\frac{1}{2}\text{ area cardioid}}$$

$$\bar{y}=\frac{\int y\,dm}{\int dm}=\frac{\int_0^\pi\int_0^{a(1+\cos\phi)} r^2\sin\phi\,dr\,d\phi}{\frac{1}{2}\text{ area cardioid}}$$

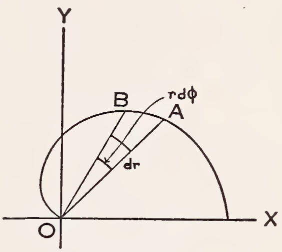


FIG. 125.

The denominator has the value

$$\int_0^\pi\int_0^{a(1+\cos\phi)} r\cdot dr\cdot d\phi=3\pi a^2/4.$$

The evaluation of the above integrals gives the results

$$\bar{x}=5a/6,\ \bar{y}=8a/9\pi.$$

Where more than one of the coordinates of the center of gravity is to be determined by integration, care should be taken to compute the denominator integral intact, without the cancellation of any numerical factor.

Example 5.—Find the center of gravity of the area of one quadrant of the ellipse $x=a\cos\phi$, $y=b\sin\phi$.

Solution.—Consider the first quadrant, described as ϕ varies from 0 to $\pi/2$ (Fig. 126). Having cancelled the constant density factor ρ, the denominator integral is one-fourth the area of the ellipse, *i. e.*, $\pi ab/4$. We shall set up the numerator integral for $\bar{x}$ in two ways, first by taking for dA a slice parallel to the Y-axis, dx wide and y long; second by taking for dA a slice parallel to the X-axis, dy wide and x long. In the first case all elements dA are equally distant from the Y-axis, and we have directly

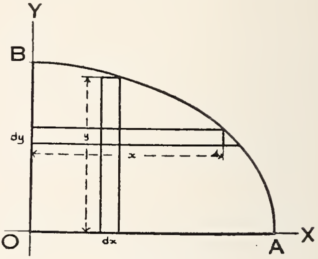


FIG. 126.

$$\int_M x\,dA = \int_M x\,y\,dx = -\int_0^{\frac{\pi}{2}} (a\cos\phi)\,(b\sin\phi)\,(-a\sin\phi\,d\phi)$$

$$= \int_0^{\frac{\pi}{2}} a^2b\sin^2\phi\cos\phi\,d\phi = a^2b/3.$$

If, however, we take a strip $dA = x\cdot dy$, we violate the restriction to which we have previously held, that all points of dm should, in the limit, be equally distant from the axis of rotation. But we have seen (Art. 58) that the moment about OY of the strip $x\cdot dy$ is equal to the moment about OY of an equal mass concentrated at the center of gravity of the strip, namely $x/2$ units from Y-axis. Hence the moment of the element $dA = x\cdot dy$ is $(x\cdot dy)x/2$, and the moment of the entire area, which is the numerator integral, is obtained by summing this to include the entire area. Hence in place of $\int_M x\cdot dA$, we have

$$\int_M (x^2dy)/2 = a^2b/2\int_0^{\frac{\pi}{2}} \cos^3\phi\,d\phi = a^2b/3.$$

Therefore,

$$\bar{x} = (a^2b/3)/(\pi ab/4) = 4a/3\pi,$$

and a similar computation will show

$$\bar{y} = 4b/3\pi.$$

Problems

155. Find the distance from the base to the center of gravity of a triangle with altitude h and base b.

156. Find the distance from the center of a circle with radius a, to the center of gravity of a sector of angle $2A$.

157. Find the center of gravity of a right circular cone with altitude h and base radius a.

158. Find the center of gravity of a hemisphere of radius a.

159. Find the center of gravity of the area bounded by the X-axis and one arch of the cycloid $x=a(\phi-\sin\phi)$, $y=a(1-\cos\phi)$.

160. Find the center of gravity of a circular arc of radius a and subtended by a central angle $2A$.

161. Find the center of gravity of the lateral area of a cone of revolution with altitude h and base radius b.

162. Find the center of gravity of the frustum of a paraboloid of revolution having a single base of radius b, the height of the frustum being h.

163. Find the center of gravity of the arc of the half arch of the cycloid $x=a(\phi-\sin\phi)$, $y=a(1-\cos\phi)$, included between $\phi=0$ and $\phi=\pi$.

164. Find the center of gravity of the area between the curves $y^2=8x$ and $x^2=8y$.

165. Find the center of gravity of a hemisphere when the density varies inversely as the square of the distance from the center. (Use a ring-shaped element of volume with cross section $r\cdot d\phi\cdot dr$.)

166. Find the center of gravity of the arc of one quadrant of the hypocycloid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

167. Find the center of gravity of the solid formed by revolving the cardioid $r=a(1+\cos\phi)$ about its axis.

168. Find the center of gravity of the area between the cissoid $y^2(a-x)=x^3$ and its asymptote.

169. Find the center of gravity of any zone of a sphere.

170. Find the center of gravity of a circle of radius a if the density varies as the nth power of the distance from a given point on the circumference. (Use polar coordinates.)

171. Find the center of gravity of one arch of the curve $y=\sin x$; of the area between this arch and the X-axis; of the surface and volume generated by revolving it about the X-axis.

172. A right circular cylinder, base radius a, axis OZ, is cut by a plane making an angle of 45° with its base and intersecting the base in OY. Find the center of gravity of that part of the surface and of the volume bounded by this plane, the XY and XZ planes.

173. Find the center of gravity of the volume obtained by rotating about the X-axis the first quadrant of the ellipse $b^2x^2+a^2y^2=a^2b^2$.

174. Find the center of gravity of the surface generated by revolving the cardioid $r=a(1+\cos\phi)$ about its axis.

61. The Center of Gravity of Composite Bodies.—The equations (4), Art. 56, may be written in the form $\int_M xdm=\bar{x}\cdot M$. The integration indicated in the left-hand member of the equation must be carried out to cover the entire body. In many problems we know the results of the integration for the parts into which the body may be divided. The value of the integral is, then, simply the sum of these results and consequently we may find the values of $\bar{x}$, $\bar{y}$, $\bar{z}$ without actually performing the integration. Thus if a body of mass M is made up of pieces having masses $M_1, M_2, \ldots M_n$ and centroids $(\bar{x}_1, \bar{y}_1, \bar{z}_1)$, $(\bar{x}_2, \bar{y}_2, \bar{z}_2)$, ... $(\bar{x}_n, \bar{y}_n, \bar{z}_n)$ respectively, and the integration for the first part gives $M_1\bar{x}_1$, for the second part $M_2\bar{x}_2$, etc., then the coordinates of the center of gravity of M, $(\bar{x}, \bar{y}, \bar{z})$, must satisfy the relations

$$\bar{x}M=\bar{x}_1M_1+x_2M_2+\ldots+\bar{x}_nM_n,$$
$$\bar{y}M=\bar{y}_1M_1+\bar{y}_2M_2+\ldots+\bar{y}_nM_n,$$
$$\bar{z}M=\bar{z}_1M_1+\bar{z}_2M_2+\ldots+\bar{z}_nM_n.$$

The following examples illustrate this procedure.

Example 1.—Find the centroid of the cross-sectional area of the steel beam shown in Fig. 127.

Solution.—For convenience take the axes as shown in the figure. Then $\bar{x}=0$ from symmetry. About any axis, moment of total area = moment AB + moment CD + moment EF, where AB denotes the area of the rectangle of which A and B are opposite vertices, etc. The centers of gravity of these rectangles have ordinates 1, 5 and 9.5 respectively. Hence, taking moments about the X-axis, we have

$$\bar{y}(12+12+18)=12\times 1+12\times 5+18\times(9.5),$$

whence $\bar{y}=243/42=5.8$ inches from the bottom of the beam.

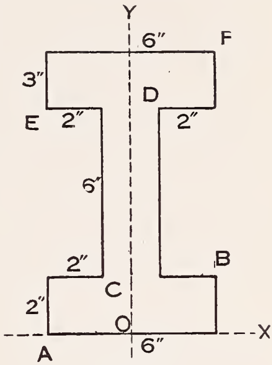


FIG. 127.

Equally well we may obtain the result by using the equation: Moment of given area = moment AF − moment BD − moment CE. Then

$$\bar{y}(66-12-12)=66\times(5.5)-12\times 5-12\times 5,$$

and therefore,

$$\bar{y}=243/42=5.8.$$

Both these solutions can be obtained from the single formula

$$\bar{y}=\frac{M_1y_1+M_2y_2+M_3y_3}{M_1+M_2+M_3},$$

if we agree that any mass added shall be given a positive sign, and that any mass cut out shall be given a negative sign.

Example 2.—Find the center of gravity of a circular area of radius 8 inches in which a hole of radius 2 inches has been cut, the center of the small circle being 5 inches from the center of the large one.

Solution.— About any axis, the moment of the large circle = moment of final area + moment of small circle. Take the line of centers as the X-axis, the origin at the center of the large circle, as indicated in Fig. 128.

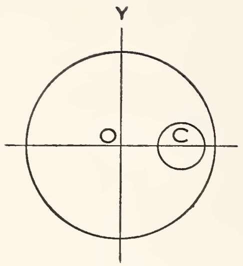


FIG. 128.

Taking moments about OY,

$$64\pi \cdot 0 = (64\pi - 4\pi) \cdot \bar{x} + 4\pi \cdot 5,$$

whence

$$\bar{x} = -\tfrac{1}{3}.$$

Or directly from the formula

$$\bar{x} = \frac{M_1 x_1 - M_2 x_2}{M_1 - M_2},$$

$$\bar{x} = \frac{64\pi \cdot 0 - 4\pi \cdot 5}{64\pi - 4\pi} = -\tfrac{1}{3}.$$

From symmetry

$$\bar{y} = 0.$$

Problems

NOTE.—These problems are to be solved without the use of integration.

175. Find the center of gravity of the trapezoid whose vertices have the coordinates (0,0), (12,0), (12,16), and (0,10).

176. A uniform plate of metal 10 inches square, has a hole of 3 square inches cut out of it, the center of the hole being 5/2 inches from the center of the plate; find the position of the center of gravity of the rest of the plate.

177. Where must a circular hole 1 foot radius be punched from a circular disk of 3 feet radius so that the center of gravity of the remainder may be 2 inches from the center of the disk?

178. The mass of the moon is 0.013 times that of the earth. Taking the earth's radius as 4000 miles and the distance of the moon's center from the earth's center as 240,000 miles, find the distance of the center of gravity of the earth and moon together from the center of the earth.

179. From a right circular cylinder of height h there is cut a cone of revolution whose base coincides with that of the cylinder, so that the center of gravity of the remaining solid coincides with the vertex of the cone. Find the altitude of the cone.

180. Find the distance above the lower base of the center of gravity of a trapezoid having bases of length a and b, and altitude h.

181. Find the center of gravity of a rifle bullet consisting of a cylinder 2 calibers in length and a paraboloid 3/2 calibers in length, having a common base. the opposite end of the cylinder containing a conical cavity 1 caliber in depth, with base equal in size to that of the cylinder. (Caliber = diameter of shell.)

182. A casting is in the form of a hollow cylinder with one end closed. The thickness of the end is 1 inch, the length of the casting 12 inches, and the radii of the inner and outer surfaces of the cylinder are 5 inches and 6 inches. Find the position of the center of gravity of the casting.

183. If the casting described in the preceding problem is filled with a material one-fourth as heavy as the material of the casting, find the position of the center of gravity.

184. A solid consists of a hemisphere of radius a and a cone of revolution, having a common base. The altitude of the cone is a. Find the distance of the center of gravity from the common base.

185. Derive the formulæ for x_a and x_b for the centers of gravity of the plane sections of beams given on pages 84-86 of *Hudson's Engineer's Manual.*

62. The Theorems of Pappus and Guldinus.—*First Theorem.*—The volume of any solid generated by the revolution of a plane area about an external axis is equal to the product of the area of the generating figure and the distance its center of gravity moves.

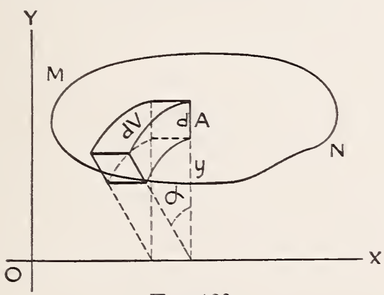


Fig. 129.

Second Theorem.—The area of any surface generated by the revolution of a plane curve about an external axis is equal to the

product of the length of the generating curve and the distance its center of gravity moves.

To prove the first theorem: Let the figure MN (see Fig. 129), with area A and center of gravity at $(\bar{x}, \bar{y})$, revolve through an angle α about OX, which does not intersect the area MN. Let dA, at the point (x, y), denote an element of area in MN, and dV the corresponding element of volume, obtained by rotating dA about OX through the distance $y\alpha$. Then

$$dV = y\alpha \cdot dA,$$

whence

$$V = \alpha \int_{MN} y\, dA.$$

But

$$\int_{MN} y\, dA = \bar{y} \cdot A,$$

therefore,

$$V = \bar{y}\alpha \cdot A,$$

which was to be proved.

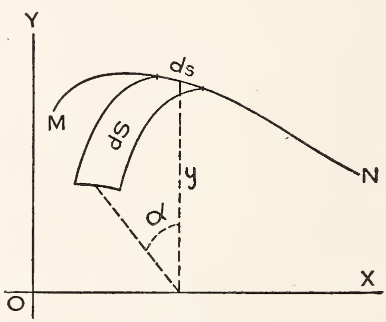


FIG. 130.

To prove the second theorem: Let the plane curve MN (see Fig. 130), of length l and center of gravity at $(\bar{x}, \bar{y})$, revolve through an angle α about OX, which does not intersect MN

except possibly at its end points. Let ds, at the point (x, y), denote an element of the arc MN, and dS the corresponding element of surface generated by rotating ds about OX through the distance $y\alpha$. Then

$$dS = y\alpha\, ds,$$

whence

$$S = \int_{MN} \alpha y\, ds.$$

But

$$\int_{MN} y\, ds = \bar{y} \cdot l,$$

therefore,

$$S = \bar{y}\alpha \cdot l,$$

which was to be proved.

Note that both theorems are valid when α has any value from 0 to 2π.

Problems

186. An equilateral triangle revolves about its base, whose length is a. Find the surface and the volume generated.

187. Find the surface and the volume of the anchor ring generated by a circle of radius a making a complete revolution about an axis b units distant from the center of the circle $(b>a)$.

188. Using the known values for the surface and the volume of a sphere, find the position of the center of gravity of the arc, and of the area, of a semicircle.

189. Find the volume obtained by revolving about its longest side a triangle whose sides are 6, 8, and 10.

190. Using the results of Ex. 5, Art. 60, find the volume of an ellipsoid of revolution.

191. Using the result of Prob. 168, find the volume generated by revolving about its asymptote the area between the cissoid and its asymptote.

63. Approximate Integration.—Hitherto we have solved only problems in which it was possible to express the function y as

8

an explicit formula in terms of x, and in which, moreover, it was always possible to find the indefinite integral of y. But many cases arise in practice for which one or the other of these conditions is not satisfied. An example of the first sort is to find the water-line area of a ship from a careful drawing when the equation of the bounding curve is not known and cannot be written in any simple form; an example of the second sort of difficulty arises in finding the quantity of water flowing per second through a circular outlet of radius r, whose center is h feet below the surface of the water. A solution of this last problem involves the evaluation of

$$\int_{-r}^{+r} \sqrt{(r^2-x^2)(h-x)}\, dx,$$

and this, called an elliptic integral, cannot be expressed in terms of the functions treated in the elementary Calculus. In such cases we must resort to approximate methods for the evaluation of the *definite integral*.

To derive an approximate formula, graph y as a function of x, obtaining B_oB_n (see Fig. 131). Then restricting y not to change sign between $x=a$ and $x=b$, we have

$$\int_a^b y\, dx = \text{area } A_oB_oB_nA_n.$$

Divide this interval from $x=a$ to $x=b$ into n equal parts $h=(b-a)/n$ in length, and at the points of division $A_0, A_1, \ldots,$ A_n, erect ordinates $y_o, y_1, \ldots, y_n$, meeting the curve in $B_o, B_1, \ldots, B_n$, respectively. The points of division must be so chosen that the values of the ordinates can be obtained from available data, by measurement or by computation. An obvious first approximation to the area A_oB_n is obtained by replacing the arcs B_oB_1, $B_1B_2, \ldots, B_{n-1}B_n$, by their chords and then adding the areas of the trapezoids $A_oB_oB_1A_1$, etc. This is some-

times used, under the name of the "Trapezoidal Rule." A better approximation can usually be obtained by taking the points B in sets of three or more and passing through each set of points a curve which, for the small distance usually taken, very nearly coincides with the corresponding piece of the given curve, and the area under which can be found by the usual methods of the Calculus.

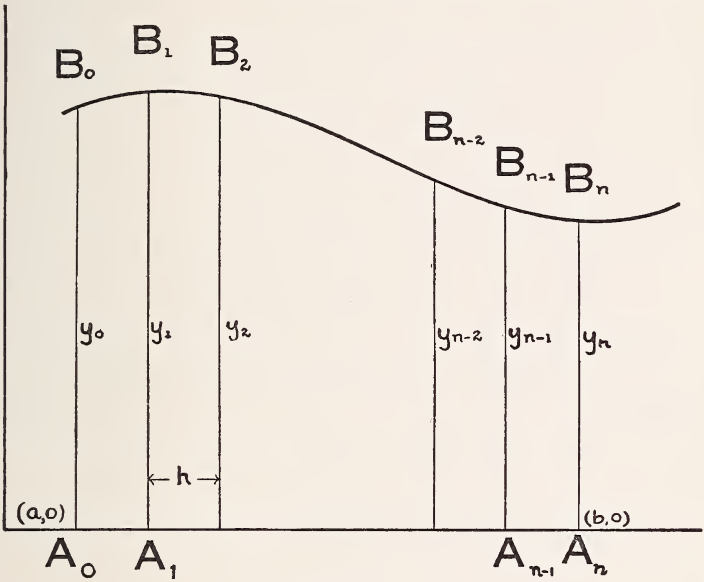


FIG. 131.

64. Simpson's Rule.—One of the most useful of these approximate formulas, known as "Simpson's Rule," requires that the number, n, of divisions of A_0A_n shall be *even*, and approximates the curve through each set of three successive points B by an arc of a cubic curve of the form

$$y=a+bx+cx^2+dx^3.$$

To find the area between each segment of the arc and the X-axis, suppose the Y-axis to be translated to coincide with the second of the three successive ordinates y_0, y_1, y_2, spaced h units apart. (See Fig. 132.)

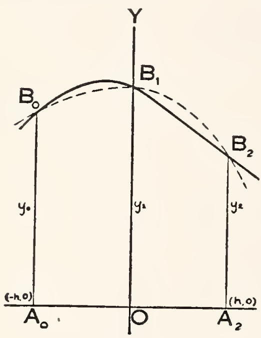


FIG. 132.

We have

$$\text{Area } A_0B_2 = \int_{-h}^{+h} y\,dx = \int_{-h}^{+h} (a+bx+cx^2+dx^3)\,dx$$

$$=2ah+2ch^3/3$$

$$=(h/3)(6a+2ch^2).$$

But

$$y_0=a-bh+ch^2-dh^3,$$

$$y_1=a,$$

$$y_2=a+bh+ch^2+dh^3,$$

whence

$$y_0+4y_1+y_2=6a+2ch^2.$$

Therefore,

$$\text{area strip } A_0B_0B_2A_2=(h/3)(y_0+4y_1+y_2).$$

Similarly,

$$\text{area strip } A_2B_2B_4A_4=(h/3)(y_2+4y_3+y_4).$$

. . . .

$$\text{area strip } A_{n-2}B_{n-2}B_nA_n=(h/3)(y_{n-2}+4y_{n-1}+y_n).$$

Adding, we obtain, as an approximate value for $\int_a^b y\,dx$,

$$(h/3)(y_0+4y_1+2y_2+4y_3+\;\ldots\;+2y_{n-2}+4y_{n-1}+y_n).$$

This is Simpson's formula. While we have used the area under a curve in obtaining this result it should be understood clearly that it is not merely a device for approximating such areas, but that it is rather a method for approximating the value of the definite integral $\int_a^b y\,dx$.

65. *Illustrative Example*

Find the height of the center of gravity above the lower base of the solid shown in Fig. 133. The solid is 24 feet high, and the five parallel sections whose areas in square feet are given, are 6 feet apart. We have to compute

$$\bar{x} = \frac{\int_0^{24} x\,A\,dx}{\int_0^{24} A\,dx}.$$

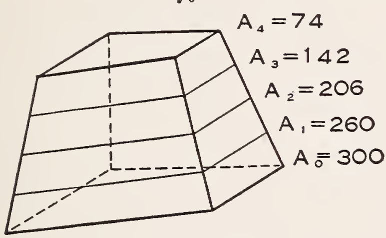


FIG. 133.

For the numerator $y = x\,A$, for the denominator $y = A$, and $h = 6$. The computation can profitably be arranged in tabular form, as follows:

A	c	cA	x/h	cAx/h
300	1	300	0	0
260	4	1040	1	1040
206	2	412	2	824
142	4	568	3	1704
74	1	74	4	296
	12	2394		3864

$$\bar{x} = \frac{\int_0^{24} x\,A\,dx}{\int_0^{24} A\,dx} = \frac{(h/3)\,(\text{sum of } cAx)}{(h/3)\,(\text{sum of } cA)}$$

$$= \frac{\text{sum of } cAx}{\text{sum of } cA} = \frac{3864 \times 6}{2394} = 9.68 \text{ feet.}$$

Instead of using values of x as multipliers in the fourth column, to lessen the computation we have used the values of x/h and afterward multiplied the sum of the last column by h.

Note that the column c reads the same up or down and its sum is $3n$. If we count the number of times each ordinate is repeated, the sum of cA is the sum of $3n$ ordinates. Referring to Fig. 131, the base equals hn, and

$$\begin{aligned}\text{area} = \int_a^b y\,dx &= (h/3)\,(\text{sum of } cy)\\ &= (hn) \times (\text{sum of } 3n \text{ ordinates})/3n\\ &= \text{base (average ordinate)},\end{aligned}$$

or again,

$$= \text{base (sum of } cy)/(\text{sum of } c).$$

Thus in the example the volume of the solid is given by

$$V = \int_0^{24} A\,dx = 24 \cdot 2394/12 = 4788 \text{ cubic feet.}$$

Problems

192. Find the area bounded by the curve $8y = x^2$, the X-axis and the lines $x = 2$ and $x = 4$. Check the result by integration. Under what conditions on y will Simpson's Rule, for $n = 2$, give the exact value of $\int_a^b y\,dx$?

193. Find the area between a segment of a curve and the X-axis if ordinates taken 4 feet apart have lengths 3.6, 4.2, 4.7, 5.1, 5.4, 5.6, 5.7 (all in feet). Find the distance of the center of gravity from the left ordinate.

194. Find the length of the perimeter of the ellipse $x = 4\cos\phi$, $y = 3\sin\phi$, taking $h = 15°$.

195. The quantity of water flowing per second through a circular orifice of radius r whose center is constantly h feet below the surface of the water is $9.76\int_{-r}^{+r}\sqrt{(r^2 - x^2)\,(h - x)}\,dx$, r and h being given in feet. Graph the integrand as a function of x, and compute the integral, when $r = 1$ foot and $h = 2$ feet.

196. A solid 20 feet high has a rectangular base 20 x 22, sections parallel to the base at intervals of 5 feet are rectangles 15 x 17, 11 x 13, 8 x 10, and 6 x 8 (all in feet). Find the volume of the solid, and the distance of the center of gravity above the base.

197. Find the area and the center of gravity of the cross section of a steel rail, 6 inches high, symmetrical about a vertical

axis and whose width in inches taken perpendicular to this axis at half-inch intervals, beginning at the base, is respectively 6, 5.5, 2.23, 1.24, 1.0, 1.0, 1.0, 1.18, 1.24, 2.5, 4.24, 4.08, 4.

198. The water-line area of a ship is 300 feet long, and half-chords of this area, measured perpendicular to its longitudinal axis at intervals of 12.5 feet, are found to be 0, 12, 18.5, 22.5, 25.5, 27, 29, 29.5, 30, 30.5, 31, 31.5, 32, 31.5, 31, 30.5, 29.5, 28.5, 27, 24, 19.5, 14, 8.5, 3, 0.5. Find the water-line area.

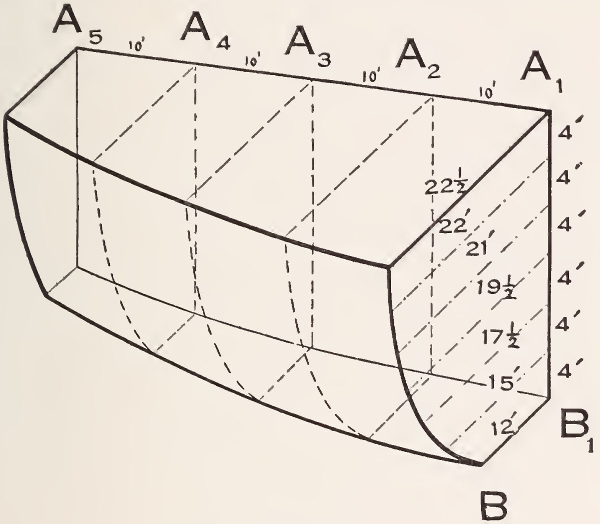


FIG. 134.

199. Cross sections of a ship, taken at intervals of 24 feet, have the areas in square feet 160, 360, 480, 553.9, 420, 140, and 0. Find the volume and the distance of the center of gravity from the stern.

200. Find the area of the section A_1, the volume of the bunker, and the distance of the center of gravity from the end section A_1B, from the data given in Fig. 134. The sections A_1, A_2, A_3, A_4, A_5 are perpendicular to A_1A_5, and the distances in A_1B are measured perpendicularly to A_1B_1.

$$A_2 = 4220 \text{ square feet.}$$
$$A_3 = 3740 \text{ square feet.}$$
$$A_4 = 3020 \text{ square feet.}$$
$$A_5 = 2020 \text{ square feet.}$$

66. Moment of Inertia.—The integral, $\int_M r^2 dm$, where r is the distance of an element of mass dm from an axis, is of such frequent occurrence in mechanics and its applications that a preliminary study of its computation is desirable. We shall show later (Art. 172) that the moment which measures the resistance that a body offers, through its inertia, to change in angular velocity, is proportional to this quantity; hence the name

$$\text{Moment of Inertia} = I = \int_M r^2 dm.$$

It follows at once from the definition that the moment of inertia of a composite body is equal to the sum of the moments of inertia of the pieces into which the body can be broken up. In particular, if the body is made up of a number of particles, the integral sign becomes a simple summation.

67. *Illustrative Examples*

Example 1.—Find the moment of inertia about one side, of a thin rectangular plate of constant density.

Let the plate have dimensions b by h. Take two intersecting sides of the plate as axes, as indicated in Fig. 135, and find I_x. Take as the element of area a slice dy thick, parallel to the x-axis. Then

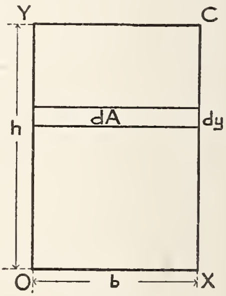


FIG. 135.

$$dm = \rho dA = \rho b\, dy.$$
$$I_x = \int_M r^2 dm = \int_0^h y^2 \rho b\, dy = \rho b h^3/3.$$

But

$$M = \rho b h,$$

therefore

$$I_x = M\, h^2/3.$$

Note that moment of inertia is measured in a hitherto unfamiliar unit having dimensions Mass $\cdot$ (length)^2; for example in pound-inch^2 units, ton-foot^2 units, etc. Thus if the rectangle of Example 1 has a mass of 20 pounds and $h=6$ inches, its moment of inertia $I_x = M \cdot h^2/3 = 20 \cdot 6^2/3 = 240$ pound-inch^2 units. In many applications we are concerned with another quantity $\int r^2 dA$. The dimensions of this are clearly (length)^4, length entering to the second degree on account of r^2 and again on account of dA. This quantity has nothing to do with inertia, but is nevertheless commonly called a "moment of inertia" and denoted by I, on account of its similarity in form to the true moment of inertia. This quantity taken for any plane area is the same as the true moment of inertia of a flat plate of unit density and unit thickness occupying the same area, for in this case we should have $dM = dA$. Thus the moment of inertia of the *area* of a rectangle 6 inches by 8 inches about the 8-inch side is $8 \times 6 \times 6^2/3 = 576$ inch^4 units.

Example 2.—Find the moment of inertia of a homogeneous thin circular plate about an axis perpendicular to the plane of the circle through its center.

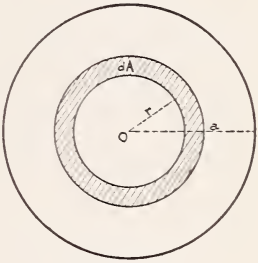


FIG. 136.

Take a ring-shaped element of area of radius r, as indicated in Fig. 136. Denote the radius of the given circle by a. Then

$$dm = \rho 2\pi r\, dr,$$

and

$$I = \int_M r^2 dm = 2\pi\rho \int_0^a r^3 dr$$
$$= \pi\rho a^4/2 = Ma^2/2.$$

Similarly the moment of inertia about its geometric axis of a homogeneous right circular cylinder having this circle as a base is also $M\,a^2/2$.

Example 3.—Find the moment of inertia of any homogeneous solid of revolution about its axis.

Take as X-axis the axis of symmetry for the solid, cutting its bases in the points $(a, 0, 0)$ and $(b, 0, 0)$. Take as the infinitesimal element of mass a cylindrical slice with radius y and

thickness dx (Fig. 137), whose moment of inertia about OX, as found in the preceding example, is

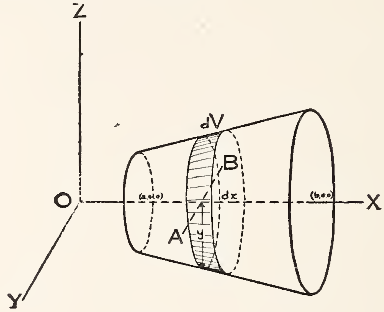


FIG. 137.

$$dI_x = \rho\,(\pi y^2\,dx)\,(y^2/2),$$

whence,

$$I_x = \frac{\pi\rho}{2}\int_a^b y^4\,dx.$$

Example 4.—Find, using Simpson's Rule, the moment of inertia about OB of the area $OBCA$ (Fig. 138), all dimensions being in feet. Here

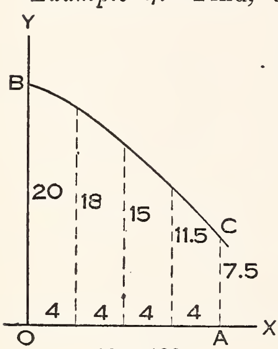


FIG. 138.

$$dA = y\,dx,\quad \rho = 1,$$

whence

$$I_{OB} = \int_M r^2\,dm = \int_0^{16} x^2\,y\,dx.$$

For Simpson's Rule, $h=4$, $n=4$, and x's have a common factor 4. Hence we have

$$I_{OB} = (4/3)\ \ 726 \times 16 = 15{,}488\ \ (\text{feet})^4.$$

$x^2/16$	y	$x^2y/16$	c	$cx^2y/16$
0	20	0	1	0
1	18	18	4	72
4	15	60	2	120
9	11.5	103.5	4	414
16	7.5	120	1	120
				726

Problems

201. Find the moment of inertia about its base of a triangle of altitude h and base b.

202. Find the moment of inertia of a right circular cone about its geometric axis.

203. Find the moment of inertia about a diameter of a homogeneous sphere of radius a.

204. Find the moment of inertia of the area bounded by the cardioid $r=a(1+\cos\phi)$, (a) about the initial line as an axis, (b) about a line perpendicular to the plane of the area, through the point $(0, \pi)$.

205. Find I_x for the area bounded by the astroid $x^{2/3}+y^{2/3}=a^{2/3}$.

206. Find the moment of inertia about its major axis of the area bounded by an ellipse with semi-axes a and b.

207. Find the moment of inertia of the area bounded by the parabola $y^2=(b^2/a)x$ and the line $x=a$, about the axis of the area; also about the tangent to the parabola at the vertex.

208. Find the moment of inertia about the X-axis of the solid obtained by revolving the area of the preceding problem about the X-axis.

68. Theorem of Parallel Axes.—When the moment of inertia of a body about any axis is known, its moment of inertia about any parallel axis can be obtained without further integration by using the following important theorem:

If the moment of inertia about any axis l, of a body whose mass is M be denoted by I_l, and its moment of inertia about a parallel axis through its center of gravity be denoted by I_G, then

$$I_l=I_G+d^2M,$$

where d is the distance between the axes.

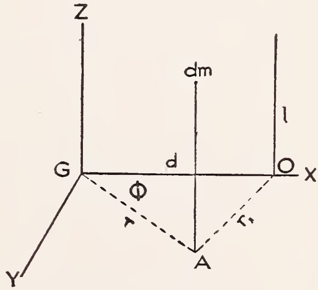


FIG. 139.

Pass a plane through G perpendicular to the parallel axes,

cutting l in O (Fig. 139). Take GO as the X-axis in a system of coordinates, with origin at G. Let A be the projection on this plane of any element of mass dm. Let $GO=d$, $GA=r$, $AO=r_1$, $OGA=\phi$. Then, using the Law of Cosines, from Plane Trigonometry,

$$I_l=\int_M r_1^2\,dm=\int_M (r^2+d^2-2\,r\,d\cos\phi)\,dm$$
$$=\int_M r^2\,dm+d^2\int_M dm-2d\int_M r\cos\phi\,dm.$$

But $r\cos\phi=x$, whence

$$\int_M r\cos\phi\,dm=\int_M x\,dm=\bar{x}M,\ \text{and}\ \bar{x}=0,$$

since G is the center of gravity for M. Further,

$$\int_M r^2\,dm=I_G,\qquad \int_M dm=M.$$

Therefore

$$I_l=I_G+d^2M.$$

Corollary: If l and l_1 are two parallel lines distant respectively d and d_1 from a parallel axis through the center of gravity of a body of mass M, then

$$I_l-d^2M=I_G=I_{l_1}-d_1^2M.$$

69. Radius of Gyration.—From its definition, moment of inertia must have the dimensions $(\text{length})^2\times\text{mass}$, and hence can be expressed in the form

$$I=k^2M.$$

This defines a length k, measured from the axis of rotation, and called the *radius of gyration*. Just as we have previously replaced the turning moment about any axis of the gravitational forces acting on a body with mass M, by the moment due to a particle endowed with the same mass and located at the center of gravity of the body; so here we replace the moment of inertia I of any

mass by that of a particle or system of particles having together the same mass and distributed in any convenient manner on the surface of a circular cylinder whose radius is k and whose axis is the axis of rotation of the given mass.

Substituting $I=k^2 M$ for I in the theorem of parallel axes, and cancelling M, we obtain

$$k_l^2=k_G^2+d^2.$$

70. *Illustrative Examples*

Example 1.—Find the radius of gyration for Example 1, and for Example 2 of Art. 67.

In Ex. 1, $I_x=M\,h^2/3$, therefore $k_x^2=I/M=h^2/3$.
In Ex. 2, $I=M\,a^2/2$, therefore $k^2=I/M=a^2/2$.

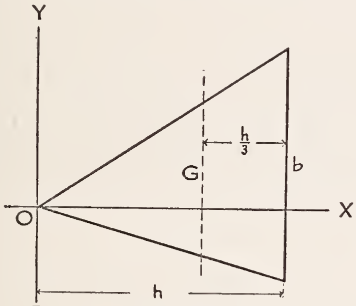


FIG. 140.

Example 2.—Find k^2 for a triangle of base b and altitude h about the base as an axis; also about a line through the center of gravity of the triangle, parallel to the base.

In solving Prob. 201, it was found that the moment of inertia of the triangle about its base is $b \cdot h^3/12$. $M=$area of triangle$=bh/2$; therefore $k_b^2=I_b/M=h^2/6$. The distance d between the base and a parallel axis through the center of gravity is $d=h/3$ (Fig. 140). Therefore

$$k_G^2=k_b^2-d^2=h^2/6-h^2/9=h^2/36.$$

Example 3.—Find the moment of inertia of a rectangle about a gravity axis parallel to one side.

Denote by I_b and I_G respectively the moments of inertia about the base of the rectangle, and about a parallel axis through the center of gravity, these axes being $d=h/2$ apart. We showed in Example 1 that $k_b^2=h^2/3$. Therefore

$$k_G^2=k_b^2-d^2=h^2/3-h^2/4=h^2/12,$$

and

$$I=k^2\cdot M=(h^2/12)bh=bh^3/12.$$

Example 4.—Find the moment of inertia of the plane section of a beam shown in Fig. 141 about an axis through its center of gravity, parallel to the shorter side.

The center of gravity is computed to be 5 inches above the base of the section. Divide the area into the rectangles bc, bd, and de. Then, about AA we have for areas bc, bd, de, respectively, $k^2=49/3$, $25/3$, $16+4/12$.

The moment of inertia of the entire section is equal to the sum of the moments of inertia of the three pieces, that is, to the sum of the products of each k^2 by the corresponding area. Hence,

$$I=(49/3)14+(25/3)10$$
$$+(49/3)6=410 \text{ inch}^4.$$

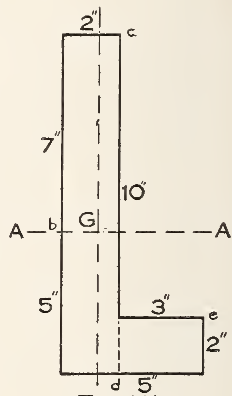


FIG. 141.

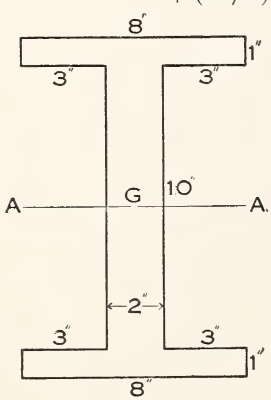


FIG. 142.

Example 5.—Find the moment of inertia about AA of the plane section of an I-beam shown in Fig. 142.

We may think of this area as that of a rectangle 8" by 12" from which two pieces, each 3" by 10", have been cut. Hence, using in each case the moment of inertia of a rectangle about a gravity axis parallel to one side, we have

$$I_{AA}=8\ 12^3/12-2(3\ 10^3/12)$$
$$=1152-500=652 \text{ in.}^4.$$
$$k_{AA}^2=652/36=18.11 \text{ in.}^2.$$

Problems

Find I and k^2 for each of the following bodies (Problems 209 to 220) about the axis indicated:

209. The Tee-section shown in Fig. 143 about AB.

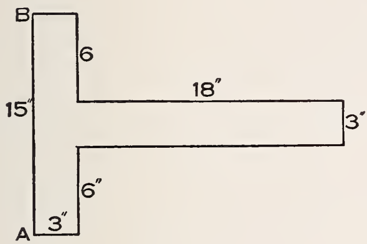


FIG. 143.

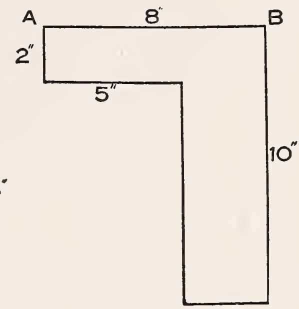


FIG. 144.

210. The angle section shown in Fig. 144 about an axis through the center of gravity, parallel to AB.

211. A hollow rectangle, outside dimensions b by h, inside dimensions b_1 by h_1, about a gravity axis parallel to the base.

212. A hollow circle, outer radius R, inner radius r, about a diameter.

213. A hollow right circular cylinder, outer base radius R, inner base radius r, about its geometric axis.

214. The one base frustum of a paraboloid of revolution about its axis.

215. A uniform rod of length h about an axis perpendicular to the rod at one end.

216. A semi-circumference about its diameter.

217. A semi-circular area about a tangent at one end of its diameter.

218. The plane sections of

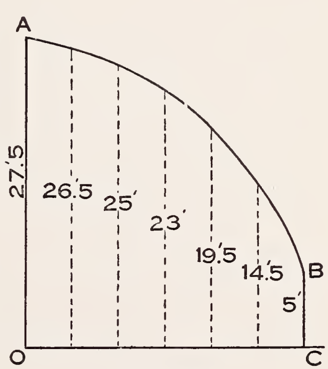


FIG. 145.

Hudson's Manual, pages 84, 85, 86, about the axes AA and BB.

219. The cross section of the steel rail shown in Example 5, Art. 70, about its base.

220. The area $OABC$ about OA, where $OC=25$ feet, and seven equally spaced ordinates, beginning with OA and ending with CB, are found by measurement to have the values 27.5, 26.5, 25, 23, 19.5, 14.5, and 5, all in feet (Fig. 145).

71. Polar Moment of Inertia.—The moment of inertia of a thin plate (or plane area) about an axis perpendicular to its plane is called its polar moment of inertia, and is denoted by I_p. Take the given axis, intersecting the given plane in O (Fig. 146), as the Z-axis in a system of rectangular coordinates, and any two perpendicular lines through O, in the given plane, as X- and Y-axes. Then

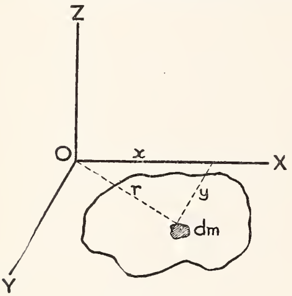


FIG. 146.

$$I_p=\int_M r^2\,dm=\int_M (x^2+y^2)\,dm=\int_M x^2\,dm+\int_M y^2\,dm=I_x+I_y.$$

Therefore

$$I_p=I_x+I_y.$$

Since OX and OY are any two perpendicular lines through O in the plane, it follows that the sum of the moments of inertia of a plane area, with respect to two rectangular axes in its plane, is the same as the sum of the moments of inertia of this area with respect to any other two rectangular axes in the same plane passing through the same point. Consequently, if I_x is the maximum value of the moments of inertia about all lines through O in the plane, I_y must be the minimum value (and *vice versa*), since their sum is constant. Two axes OX and OY having this property, that I_x is a maximum and I_y a minimum, are called *principal axes* for the point O, and I_x and I_y are called *principal moments of inertia* for O.

It can be shown that an axis of symmetry for the area, when it exists, is a principal axis. Further, if we take this axis of symmetry for OX, and for OY the perpendicular axis through the center of gravity of the area, then the smaller of I_x and I_y will be the least moment of inertia that can be obtained for the area about any axis in its plane.

72. *Illustrative Examples*

Example 1.—Find the moment of inertia of the area of a circle about a diameter, and about a polar axis through a point on the circumference.

It was shown in Ex. 2 of Art. 67, that the polar moment of a circle about an axis through its center is $Ma^2/2$. Hence the polar moment about a parallel axis through a point on the circumference (a units away) is

$$I_{p_1} = Ma^2/2 + Ma^2 = 3Ma^2/2.$$

Because of the symmetry of the circle, its moment of inertia I_D about a diameter is the same for all diameters. Hence if we take a set of axes in the plane of the circle, with origin at its center, then

$$I_x = I_y = I_D.$$

But

$$I_p = I_x + I_y = 2\, I_D,$$

and

$$I_p = Ma^2/2.$$

Therefore

$$I_D = Ma^2/4.$$

Example 2.—Find the moment of inertia of a homogeneous solid of revolution about an axis perpendicular to its axis of rotation.

Take the axis of rotation for the solid (Fig. 147) as the X-axis, and find I_y. Take as the element of mass dm a slice dx thick perpendicular to the X-axis. Then $dm = \rho\pi y^2 dx$. Let AB, parallel to OY, be a diameter of the element of volume. Then, from the preceding example,

$$d(I_{AB}) = (\rho\pi y^2 dx) y^2/4.$$

From the theorem of parallel axes,

$$d(I_y) = d(I_{AB}) + x^2 dm = \rho\pi y^2 (y^2/4 + x^2)\, dx.$$

I_y is obtained by integrating this between the proper limits.

Example 3.—Find the moment of inertia about its axis of a right circular cylinder four feet in diameter, weighing 160 pounds.

Problems of this type illustrate the importance of expressing the moment of inertia of homogeneous solids in the form $I = Mk^2$.

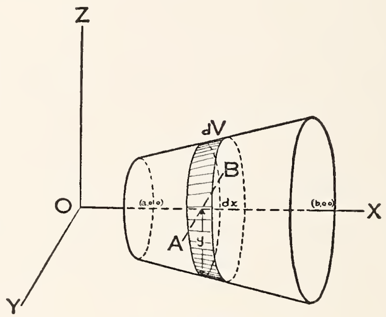


FIG. 147.

Thus, for the cylinder in question, $k^2 = a^2/2 = 2$, and $I = 2M$, where the value of M will depend on the units of mass being used in a discussion in which this problem might arise. For use in most problems in the dynamics of a rigid body it will be found convenient to use (see Art. 122) the "engineer's unit of mass" or "g-pound," which requires that a body whose weight is W pounds have a mass $M = W/g$, where g is a constant for all bodies at the same place on the earth's surface. For the purposes of the present chapter we shall take $g = 32$; it will be seen later that this requires that time shall be expressed in seconds, and length in feet in any equation in which this determination of mass enters. Thus for the present problem, $M = 160/32 = 5$ "g-pounds," and $I = 5 \times 2 = 10$ g-pound feet^2.

Problems

Find I and k^2 for each of the following bodies about the axis indicated.

221. A cube about a gravity axis parallel to an edge.

222. A rectangular prism about a gravity axis parallel to an edge.

223. An anchor ring obtained by rotating a circle of radius r so that its center traces the circumference of a circle of radius R; about the axis of rotation; about a diameter perpendicular to this.

224. A right circular cone with altitude h and base radius a, about an axis through the vertex parallel to the base; about a gravity axis parallel to the base.

225. The area of the lemniscate $r^2 = a^2 \cos 2\phi$ about a polar axis through its center.

226. The area between the parabola $y^2 = a(a - x)$ and the circle $x^2 + y^2 = a^2$ about a polar axis through the origin.

227. The head of a projectile is formed by the revolution of a semi-parabola about the ordinate b, so that the height h is the radius of the base, or one-half the caliber a; about the axis of rotation.

228. A body consisting of a hemisphere and a cone of revolution of the same base, and height equal to the radius; about an axis through the vertex, parallel to the common base.

229. The ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, about the X-axis.

230. A circle of radius a, the density varying as the square of the distance from the center; about a diameter.

231. The plane section of a beam is a rectangle 6 inches by 10 inches. Find its least moment of inertia.

232. The plane section of a column is an ellipse with semi-axes 6 inches and 8 inches. Find its least moment of inertia.

233. Find the moment of inertia about a diameter of a homogeneous sphere 18 inches in diameter, and weighing 96 pounds.

234. Find the moment of inertia about its axis of a hollow cylindrical drum, weighing 120 pounds, and having an inner radius of 18 inches, an outer radius of 24 inches.

CHAPTER VI

FLUID PRESSURE

73. Properties of Fluids.—Up to this point we have been concerned only with rigid bodies, *i. e.,* bodies which do not change their shape or volume when acted upon by external forces. This is largely an assumption, since in reality, if sufficient force is exerted, the shape or volume, or both, of any body will change. However, for practical purposes, it has been necessary and sufficiently accurate for us to assume that these bodies were rigid.

Matter exists in the form of either a solid, a liquid, or a gas, although sufficient change in temperature and pressure will cause any substance to pass from one of these forms to another. A solid has a definite form and volume and offers marked resistance to any change of these; a liquid has a definite volume but takes the form of the vessel containing it; a gas has neither definite form nor volume but expands till it fills the vessel containing it.

We define a fluid as a body which offers no permanent resistance to forces which tend to change its shape.

A "perfect fluid" is frictionless, and can exert *no* resistance to shear or tensional stress. (See chapter on "Deformable Bodies," Arts. 84 and 89.) Such a fluid takes the exact form of a containing vessel. Actually, fluids do offer a slight resistance to distortion of form, and the measure of this resistance is the "viscosity" of the fluid.

Water at rest, which is the case in Hydrostatics, may be considered, with negligible error, as a perfect fluid.

Fluids can, broadly speaking, be divided into two general classes, *liquids* and *gases.*

A *liquid* is a fluid which changes volume very slightly under compression.

A *gas* is a fluid whose volume changes sensibly under compression.

All fluids have weight. In the case of liquids in an open vessel the thing of importance is the force due to the weight of the liquid, and its effect upon the walls of the container or upon the surfaces of submerged bodies. On the other hand, a gas confined in a closed vessel completely fills the vessel, and the forces acting upon the walls of the vessel are due primarily to the pressure of the gas under compression, and not to the weight of the gas, the force produced by the latter being relatively negligible.

74. Pressure.—*Theorem.*—" When a fluid is at rest, the force which it exerts on any surface with which it is in contact, is normal to that surface."

For, if the force were not entirely normal it would have an unbalanced component tangent to the surface, and, since the fluid cannot resist this tangential force, motion would take place, which is contrary to hypothesis. Thus, in the Fig. 148 is rep-

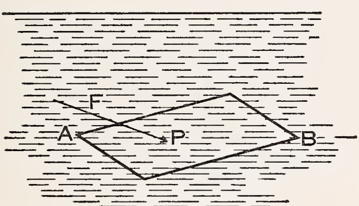


FIG. 148.

resented a plane AB submerged in a liquid. If the force F at the point P were not normal, *i. e.*, not perpendicular to the plane, there would be an unbalanced component of F in the direction parallel to the plane which could not be resisted by the fluid.

Liquids in an open vessel present a " free surface " to the air. An immediate consequence of the above theorem is " that the free surface of a liquid at rest is perpendicular to the force of gravity, and so is practically a plane."

In considering the forces exerted by liquids with a free surface, we shall neglect the force due to the weight of the air on this surface, unless otherwise specified.

If a plane surface be submerged horizontally in a liquid, to a depth of h feet below the free surface, then on every square foot of this plane there is a force exerted normal to it, of

$$(hw) \text{ pounds,}$$

where w is the weight per cubic foot of the liquid. The distance h is called the "head" on this plane. This result rests on the assumption that the density of the liquid is constant. For the small depths usually encountered in hydrostatic problems, this assumption produces only a negligible error. If the area of the horizontal plane is A square feet, then the total force F, exerted by the liquid on the plane, is

$$F = Ahw \text{ pounds.}$$

The intensity of the force is uniform over the horizontal plane surface, *i. e.*, is the same at every point. If F is the total force, and A the area, then

$$p = F/A$$

is defined as the "pressure" on the horizontal plane, and is given in pounds per square foot, or some similar units. In the case of the horizontal plane, the pressure is the same at all points of the plane.

75. *Theorem.*—"The pressure (force per unit area) at any point of a fluid at rest, is the same in all directions."

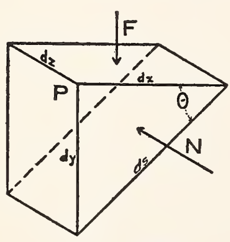


FIG. 149.

Consider a portion of the fluid, in the form of a right triangular prism (Fig. 149), the legs of the right angle being horizontal and vertical as in the figure. Let F be the force on the horizontal face. Then

$$\frac{F}{dx \cdot dz} = p, \text{ or } F = p \cdot dx \cdot dz,$$

where p is the pressure on that face. As dx and dz approach zero, p remains constant. This force F must be balanced by the vertical component of N, the force

normal to the inclined face, since on all the other faces the forces have no vertical components. This vertical component is $N \cos\theta$, and hence

$$F = N \cdot \cos\theta.$$

Let p' be the pressure on the inclined face at its upper edge. This pressure increases as we go down the inclined face, due to the increasing heads on this surface. If ϵ is the increase in pressure at the lower edge, then

$$p' \cdot dz \cdot ds < N < (p' + \epsilon) \cdot dz \cdot ds.$$

But

$$ds \cdot \cos\theta = dx,$$

and so

$$p' \cdot dx \cdot dz < F < (p' + \epsilon) \cdot dx \cdot dz$$
$$\therefore p' < p < p' + \epsilon.$$

Now as the prism shrinks to a point P by letting dx, dy, and dz approach zero, then $\epsilon \doteq 0$, and so, at the limit,

$$p' = p.$$

Resolving horizontally, parallel to dx, we find the limit of the pressure on the face $dy\ dz$, the same as that on the inclined face, and so is equal to p. Since θ may be any angle, this proves that the pressure at any point is the same in all directions.

76. If we now consider a plane of finite extent, submerged and inclined to the horizontal, the pressure at a point P, at depth h, is then the same as if the plane were turned to a horizontal position at depth h, and so

$$p = F/A,$$

where here F is the force exerted on any area A at constant depth h. In the case of the inclined plane, p varies from point to point on the plane, but satisfies the relation

$$p = wh.$$

In general, if any surface be submerged, the pressure at any point at depth h is given by

$$p = wh,$$

and the force producing this pressure is normal to the surface at each point.

77. The Transmissibility of Pressure.—Suppose a fluid completely fills a closed vessel, and a constant force be applied to a piston in the walls of the vessel (Fig. 150). If the force tends to push the piston in, the fluid becomes compressed. Since the fluid and the piston are at rest, the force on the outside produces a definite pressure on the fluid immediately adjacent to the piston head, which pressure must be transmitted without change to every point on the inside of the walls of the vessel. To prove this statement, assume that the force on the piston is the only force acting on the fluid, and consider a right circular cylinder of the fluid extending normally from a unit area at A on the piston to a point P in the fluid. The forces acting on the curved surface of this cylinder are normal to that surface and so have no com-

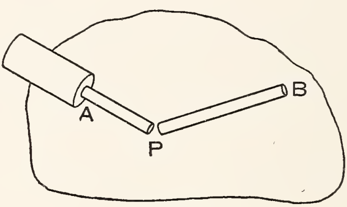


FIG. 150.

ponent along the axis of the cylinder. Since the cylinder is at rest, then the forces acting normal to its ends are in equlibrium, and so the pressure at P is the same as that at A. Now, since the pressure at P is the same in all directions, we may turn the cylinder about P so as to end at or near any point B of the walls of the vessel, and so conclude that the normal pressure at B is the same as that at P and so is the same as that at A.

This pressure is of course in excess of that due to the weight of the fluid, the latter varying with the depth.

The principle involved in the above discussion is called that of "the transmissibility of pressure within a fluid." The hydrostatic press is an application of this principle.

Example.—The lever of a hydrostatic press is 3 feet long (Fig. 151). The distance from its fulcrum to where it acts on a

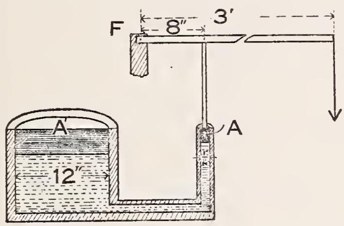


FIG. 151.

small piston is 8 inches. The diameters of the pistons are $\frac{1}{2}$ inch and 1 foot. If 1 pound is applied at the end of the lever, what weight (W) will the large piston just lift?

Here the effective force at the small piston is found, by taking moments, to be

$$36/8 \text{ pounds.}$$

Hence the pressure produced in the water is

$$p = \frac{4.5}{\frac{\pi}{16}(=A)} \text{ lbs./sq. in.} = \frac{W}{36\pi(=A')};$$

$$\therefore W = 2592 \text{ lbs.}$$

78. Total Pressure Force.—We have seen that the pressure at a point of a submerged surface is given by

$$p = wh.$$

If the surface is horizontal, then the pressure is constant, and the total force, F, is given by

$$F = whA \text{ lbs.}$$

In the general case the surface is not horizontal, and the pressure varies from point to point. If the surface is plane, the force

vectors at each point are parallel, and so the total force has a physical meaning. It is the thrust on the plane due to the liquid pressure, or the total pressure force, and it is normal to the plane. Its point of application will be considered later.

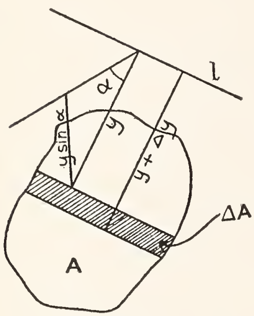


FIG. 152.

Consider any plane area A (Fig 152), submerged at an angle α with the horizontal, and let l be the line of intersection of the plane of A and the free surface. Take an element of area ΔA, in the form of a strip whose edges are parallel to the surface of the liquid, and at distances

$$y \text{ and } y+\Delta y$$

from l. Then the element of force on ΔA is $w \cdot y \sin \alpha \cdot \Delta A$ pounds; so the total force on A is given by

$$F = \lim\ (\Delta y \doteq 0) \Sigma_A w \cdot y \sin \alpha \cdot \Delta A = \int_A w \cdot y \sin \alpha \cdot dA,$$

where the integration covers the area A. But we observe here that

$$F = \int_A w \cdot \sin \alpha \cdot y dA = w \cdot \sin \alpha \int_A y \cdot dA = w \cdot A \cdot (\bar{y} \cdot \sin \alpha),$$

where $\bar{y} \cdot \sin \alpha$ is the distance of the center of area A below the surface of the liquid, or $\bar{y}$ is the distance from l, in the plane A, to the center of area. Hence we have the

Theorem.—"The total pressure force on a submerged plane area is equal to the product of the area, the weight per unit volume of the liquid, and the depth of the center of area."

Examples

1. Find the thrust on a vertical circle of radius 2 feet, submerged in water with its center 10 feet below the surface.

Here $w = 62.5$ lbs./cu. ft. $A = 4\pi$ sq. ft. and $\bar{y} \cdot \sin \alpha = 10$ ft.

$$\therefore F = (62.5) \cdot (4\pi) \cdot (10) = 7860 \text{ lbs.}$$

2. Compute the total pressure force on a vertically submerged parabolic segment, of width $2h$ feet and depth b feet, with upper edge in the surface of the liquid (Fig. 153).

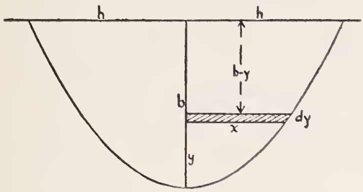


FIG. 153.

From the figure

$$F=\int_0^b w\cdot(b-y)\cdot 2x\cdot dy.$$

But here

$$x^2:h^2=y:b;$$

$$\therefore F=\frac{2wh}{\sqrt{b}}\int_0^b (by^{\frac{1}{2}}-y^{\frac{3}{2}})\,dy=\tfrac{8}{15}whb^2 \text{ lbs.}$$

3. A rectangle $h\times b$ feet is submerged with its edge h in the surface, and its plane turned $30°$ from the vertical. Determine the total pressure force on this rectangle.

Here $A=bh$, $\alpha=60°$, and $\bar{y}=\frac{b}{2}$. Hence $F=\frac{1}{4}wb^2h\sqrt{3}$ pounds.

79. Cylindrical Surfaces.—We will now consider the total pressure force in a given direction on cylindrical surfaces. A problem of practical importance is to determine the stresses set up in the material of a steam pipe or boiler due to the internal pressure force. In these cases the surfaces are cylindrical.

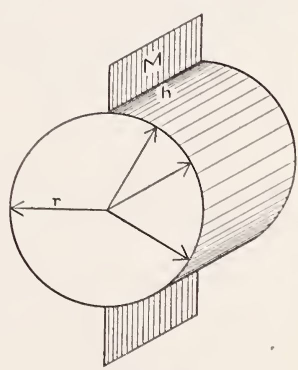


FIG. 154.

Consider first a right circular cylinder (Fig. 154) placed with axis horizontal, and full of liquid. At any point of the surface the force is normal to the surface. Since the normal forces at various points on

this surface have different directions, the total pressure force as the sum of the normal forces has no physical significance. We can, however, find the total component of pressure force in any one direction, since the force vectors so involved are all parallel.

Pass a vertical plane M through the axis of the cylinder. Then from *symmetry,* both of forces and areas, the sum of the horizontal components of the forces on the right-hand cylindrical surface is equal and opposite to that on the left-hand cylindrical surface, and these are both equal numerically to the total pressure force on each side of the vertical plane M. Hence the thrust T, to the right and left on the half cylindrical surfaces, is given by (area) $\times$ (w) $\times$ (depth of center of area), or

$$T = 2rh \cdot w \cdot r = 2wr^2h \text{ lbs.}$$

Suppose the same cylinder is full of steam at a pressure of p pounds per square inch. Here the density is negligible, and so the normal force can be taken as constant in magnitude at all points of the inner surface. In this case the thrust on the vertical diametral plane is simply the area times the pressure, and so

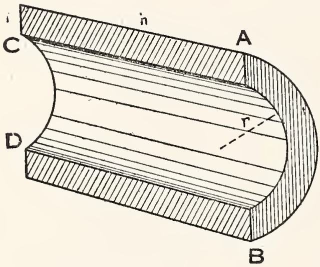


FIG. 155.

$$T = 2rh \cdot p \text{ lbs.},$$

and this is constant for any diametral plane, vertical or otherwise. This force must be sustained by the material of the cylinder, *i. e.,* the two areas AC and BD (Fig. 155) are subjected to a stress of p' pounds per square inch, given by

$$2p'ht = 2phr,$$

i. e.,

$$p' = pr/t \text{ lbs. per sq. in.}$$

Applications of this result will be given in a later chapter (Art. 89).

Problems

235. What is the pressure 40 feet below the surface of a lake?

236. A jar in the form of a frustum of a right circular cone, has a diameter of 14″ at the top and 10″ at the bottom. It is filled with 9″ of mercury and 8″ of oil. If the specific gravity of oil is .915, and 1 cubic inch of mercury weighs $\frac{1}{2}$ pound, find the total force on the bottom of the jar.

237. A 6′ square is submerged vertically in salt water until its upper edge is 5′ below the surface, and then it is turned about its horizontal upper edge through an angle of 60°. Find the force on the square.

238. A triangle is submerged in liquid with its base in the surface. Where must a horizontal line be drawn across it, so that the forces due to the liquid pressure on the two parts shall be equal?

239. If a full cylinder is held with axis horizontal, determine the resultant force acting vertically on the lower half of the curved surface.

240. Find the total pressure force on the vertical face of a semi-elliptical dam 100′ long and 15′ deep.

241. Find the force on a vertically submerged trapezoid of altitude 4′ and bases 12′ and 9′, the long edge being in the surface of fresh water.

242. If the pressure in the accumulator of a hydraulic forging press is 2 tons per square inch, and the diameter of the press cylinder is 80″, what is the force of the press?

243. Find the force on the face of a dam whose submerged vertical surface has the shape of an inverted cycloid of depth $2a$.

244. The pressure in a water pipe in the basement is 74.5 pounds per square inch, while in the fifth story it is only 48 pounds per square inch. Find the difference in elevation.

245. Find AP so that the force on K will be three times that on H (Fig. 156).

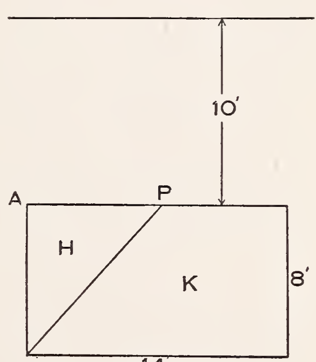


FIG. 156.

246. A board 2′ wide at one end and 2′6″ at the other, is 8′ long. Find the force on the board when placed vertically in water with the narrow end in the surface.

247. The back of a dam has a slope of 3 to 2. Find the horizontal force per linear foot upon it when the water is 13 feet deep.

248. What head of water will burst a pipe of 24″ internal diameter and $\frac{3}{4}$″ thick, the tensile strength of the material being 20,000 pounds per square inch?

249. An equilateral triangle is immersed vertically in water, with two vertices at a depth of 6′ and the third at a depth of 9′. What is the force on it?

80. Center of Pressure.—The forces producing pressure at points of a submerged plane area are all normal to the plane and so constitute a set of parallel forces. Their resultant is then a force parallel to these acting at the center of force. In hydrostatics this center of force is called the "center of pressure" (C. P.).

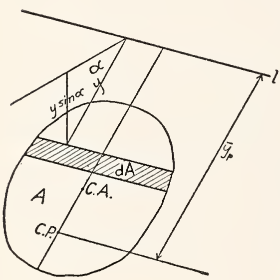


FIG. 157.

Take any submerged plane area A (Fig. 157), making an angle α with the horizontal, and let l be the line of intersection of the plane of A and the free surface of the liquid. Measure y in the plane of A, from l to a representative area element dA. Then the head on this element is $y \sin \alpha$, and the element of force is $wy \sin \alpha \cdot dA$. Let $\bar{y}_p$ be the distance from l to the center of pressure, measured in the plane of A. Then if F is the total pressure force on A, we have, taking moments about l,

$$\bar{y}_p \cdot F = \int_A y \cdot (wy \sin \alpha \cdot dA),$$

the integration to cover the area.

If $\alpha=0$, the plane is horizontal, and there is no line l. Consequently this result is valid for $\alpha\neq 0$. But in the case of a horizontal plane, the center of pressure is obviously the center of area.

If A has a line of symmetry, s, perpendicular to l, then both the center of pressure and the center of area lie on s, because of the equal forces and equal areas to right and left of s. The position in A of the center of pressure is then determined by $\bar{y}_p$ and s. If there is no line of symmetry perpendicular to l, moments can be taken about any line perpendicular to l in A, giving the horizontal distance from such a line to the center of pressure.

Excluding the horizontal case, we then have,

$$\bar{y}_p=\frac{\int_A wy^2 \sin\alpha \cdot dA}{F}=\frac{w\sin\alpha\int_A y^2dA}{w\sin\alpha\int_A ydA}=\frac{\int_A y^2dA}{\int_A ydA}=\frac{I_l}{A\bar{y}}=\frac{I_l}{S},$$

where I_l is the moment of inertia of the area A about l, $\bar{y}_g$ is the distance from l in the plane of A to the center of area, and S is the statical moment of A about l. Since

$$I_l=A\cdot k_l^2$$

we get

$$\bar{y}_p=\frac{k_l^2}{\bar{y}_g},$$

where k_l^2 is the squared radius of gyration of the area A about the line l. If k_g^2 is the squared radius of gyration of A about a line through the center of area parallel to l, we have

$$\bar{y}_p=\frac{k_g^2+\bar{y}_g^2}{\bar{y}_g}=\bar{y}_g+\frac{k_g^2}{\bar{y}_g}.$$

81. *Examples*

1. Find the center of pressure on a circular bulkhead of radius 2 feet, situated vertically with its center 12 feet below the surface of the water (Fig. 158).

Here

$$\bar{y}_p = \frac{k_g^2 + \bar{y}_g^2}{\bar{y}_g} = \frac{\frac{2^2}{4} + (12)^2}{12} = 12\tfrac{1}{12}\text{feet} = 12'1'';$$

i. e., the center of pressure is on a vertical diameter, 1 inch below the center of the circle

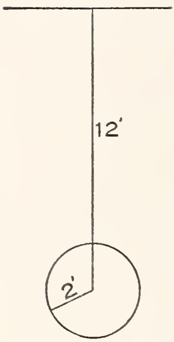


FIG. 158.

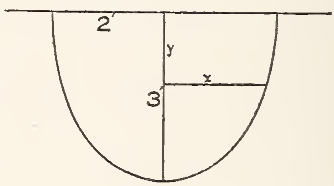


FIG. 159.

2. Find the center of pressure on a semi-ellipse (Fig. 159), submerged vertically, with minor axis in the surface, the semi-axes being 2 feet and 3 feet.

Here

$$\bar{y}_p = \frac{\int_0^3 y^2 \cdot 2x dy}{\int_0^3 y \cdot 2x dy}.$$

But

$$x^2 = \tfrac{4}{9}(9 - y^2),$$

hence

$$\bar{y}_p = \frac{\int_0^3 y^2 \sqrt{9 - y^2} \cdot dy}{\int_0^3 y \sqrt{9 - y^2} \cdot dy}.$$

The denominator integral is of the "power form" and the

numerator is easily integrated after making the substitution $y=3\sin\theta$. The result is

$$\bar{y}_p=\frac{9\pi}{16}=1.77 \text{ feet down on the major axis.}$$

82. *General Illustrative Examples*

We shall now apply the principles that have been developed to various problems in fluid pressure.

1. A vertical rectangular masonry dam is 4 feet thick and weighs 140 pounds per cubic foot (Fig. 160). Determine its height, that the pressure force exerted by a full head of fresh water against one side of it may make it fail by sliding, given the coefficient of friction, $\mu=.75$; and also the height at which it will fail by turning over.

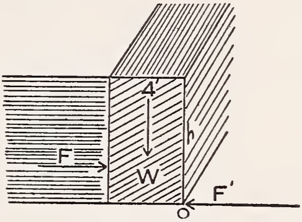


FIG. 160.

Take a foot length of the dam. Its weight is

$$W=4(1)h(140)=560h \text{ lbs.}$$

The resistance offered to sliding is then

$$F'=.75W=420h \text{ lbs.}$$

The total pressure force is

$$F=h(1)\cdot(h/2)w=125h^2/4 \text{ lbs.}$$

We must have $F=F'$, and so

$$\frac{125h^2}{4}=420h,$$

or

$$h=13.44 \text{ ft.}$$

To be on the point of overturning about O, we must have the moments of F and W equal about O. F acts at the center of pressure, which is at height $h/3$ feet above O. W acts at the center of gravity, and so has an arm of 2 feet. Hence,

$$125h^2/4\cdot(h/3)=560h(2),$$

or

$$h=10.37 \text{ feet.}$$

2. The fin keel of a yacht is shown in Fig. 161. Find the total pressure force, the depth of the center of pressure, and the

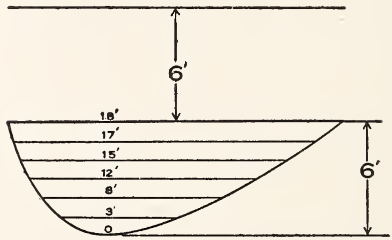


FIG. 161.

moment of the force which tends to turn the keel about the line of intersection of its plane with the surface of the water.

Here, using Simpson's Rule with $h=1$, we have the following tabulated computation:

y	x	c	xyc	$x(xyc)$
18	6	1	108	648
17	7	4	476	3332
15	8	2	240	1920
12	9	4	432	3888
8	10	2	160	1600
3	11	4	132	1452
0	12	1	0	0
			1548	12840

$$\therefore F=w\cdot\frac{h}{3}\cdot\Sigma xyc=\frac{w}{3}(1548)=516w=32250 \text{ lbs.}$$

(Where $w=62.5$ for fresh water.)

$$S=w\cdot\frac{h}{3}\cdot\Sigma x^2yc=\frac{w}{3}(12840)=4280w=267500 \text{ lb.-ft.}$$

$\bar{x}_p=4280/516=8.294$ feet below the surface.

3. An automatic water gate may be used for draining a marsh, and is so constructed (Fig. 162) that it opens when the head of water on the marsh side exceeds that on the ocean side, and closes

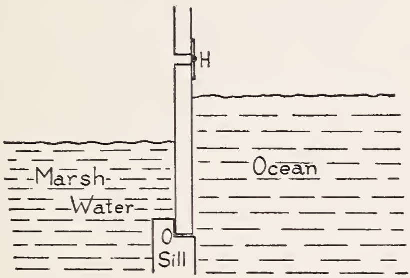


FIG. 162.

when the condition is reversed by the rising of the tide. The difference in the head, when closed, causes horizontal forces to act on the gate, at the contacts O and H. These two forces, together with the two pressure forces, form a system of four parallel forces in equilibrium (Fig. 163). Taking moments about O at the sill, we get the reaction R at the hinge H. What is the reaction, for a gate 4 feet high and 5 feet wide, when the ocean stands 3 feet above the sill, and the marsh (salt) stands 1 foot above the sill? Taking moments about O gives

$$P_1(1/3)+R(4)=P_2\cdot(3/3),$$

where

$$P_1=(1)\ (5)(64)(1/2)\text{ lbs.}$$

(the density of salt water being 64 pounds per cubic foot), and

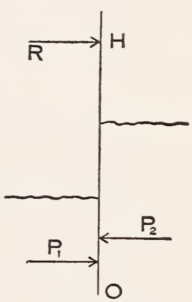


FIG. 163.

$$P_2=(3)(5)(64)(3/2)\text{ lbs.,}$$

from which

$$R=346\tfrac{2}{3}\text{ lbs.}$$

Problems

250. Find the C. P. of a rectangle submerged vertically, with its edge in the surface.

251. Find the C. P. of a vertically submerged triangle when (a) one side is in the surface, and (b) one side is parallel to the surface and the opposite vertex is in the surface.

252. Find the C. P. of a vertically submerged circular quadrant with one edge in the surface.

253. Find depth of C. P. of a vertically submerged semi-ellipse having its major axis in the surface.

254. A parabolic segment of base $2b$ and height h is submerged vertically. Find the C. P. when (a) its base is in the surface, and (b) its base is parallel to the surface and its vertex is in the surface.

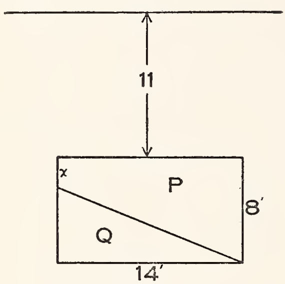


FIG. 164.

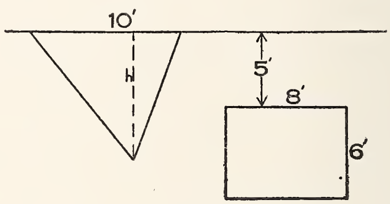


FIG. 165.

255. Find the C. P. of a submerged trapezoid, having bases a and b and altitude h, if base a is in the surface.

256. A pair of flood gates are swung on hinges 3/2′ from top and bottom of the gates which close an opening 10′ wide by 6′ deep. What is the force on each hinge when fresh water stands at the top of the gate?

257. Find the C. P. of a vertically submerged circle, if the head on its center is 2 diameters.

258. If the head on one side of a tide gate is 7′ and on the other 4′, find the resultant force and its point of application.

259. A vertical masonry dam is 4′ thick, 18′ high, and weighs 125 pounds per cubic foot. How high can fresh water stand against the dam before it overturns?

260. A cubical tank is 3′ on an edge. One vertical face is hinged at the top. The tank is filled with fresh water. What is the least force which will keep the hinged face in place, and where should it be applied?

261. The partition wall between the compartments of a settling basin is 12′ high. Water is within 1′ of the top on one side, the other side being empty. If the masonry weighs 160 pounds per cubic foot and 60 per cent of its total weight represents its ability to resist sliding, how thick should the wall be?

262. Find x so that the force on P is 50 per cent greater than that on Q (Fig. 164).

263. Triangle and rectangle are vertically submerged. Find the altitude of the triangle so that the force on it shall be 50 per cent greater than that on the rectangle (Fig. 165).

264. A dam whose cross section is a right triangle of base 4′ and altitude 15′ weighs 125 pounds per cubic foot. How high can salt water stand against the vertical face before it will turn over?

265. A rudder in the form of a semi-parabolic segment (see Fig. 166) has AB vertical, and A at a depth of 3′. Find the total pressure force and its point of application.

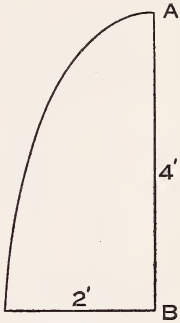


FIG. 166.

Review Problems

266. Find force and C. P. on a vertically submerged circle tangent to the surface.

267. Find force and C. P. on trapezoid with bases 10′ and 8′ and altitude 3′ submerged vertically in fresh water with 10′ edge in the surface.

268. Find the moment of the force on a vertically submerged circle tangent to the surface, about the tangent line in the surface.

269. Find force and C. P. on a rectangle $6' \times 9'$ submerged vertically in salt water, having short side horizontal and upper one 5′ below the surface.

270. If the centers of two circular pistons, of 4″ and 12″ diameter are at the same level, and the smaller one is pushed

inwards with a force of 100 pounds, how much force must be applied to the larger one to keep it from moving?

271. What is the pressure 1000 fathoms deep in the sea?

272. A cubical vessel is filled with liquid. Compare the forces on its bottom and sides.

273. A vertical water gate 40′ wide has fresh water standing 20′ deep on one side. How deep must salt water stand on the other side, in order that the forces may be equal?

274. Find the C. P. for a rectangle $a \times b$ vertically submerged, with one vertex in the surface, and a diagonal horizontal.

275. On one side of a sluice gate, water stands 8′ deep, and on the other side 4′ deep. The gate is 6′ wide and 10′ deep. What is the force on the hinges, and where is the C. P.?

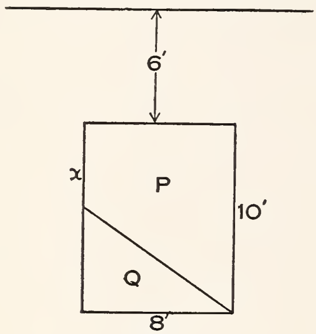


FIG. 167.

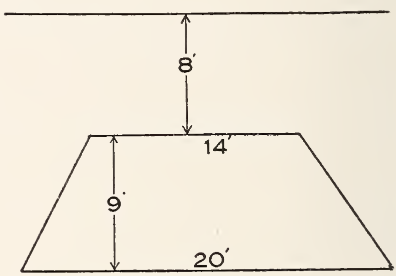


FIG. 168.

276. Find the magnitude and point of application of the force on a vertically submerged rectangle $4' \times 6'$ with short side in the surface.

277. A solid rectangular masonry dam is 6′ thick and weighs 136 pounds per cubic foot. Find its height in order that it will just be on the point of overturning, when fresh water stands against one face at a depth of $\frac{3}{4}$ the height of the dam.

278. A dam whose cross section is a triangle has a vertical back, is 3′ wide at the base and 15′ high. Find the height to which the water may rise behind it, in order to cause failure (a) by sliding, and (b) by turning, using 0.75 as the coefficient of friction and 140 pounds per cubic foot for weight of masonry.

279. What is the force on the hinges of an automatic water gate 5′ high and 8′ wide, when the sea stands 4′ above the sill on one side, and salt marsh water 2′ above the sill on the other side?

280. Find the magnitude and point of application of the force on a vertically submerged triangle of base 2′ and altitude 3′ when (a) the vertex is in the surface and the base is horizontal, and (b) the base is in the surface.

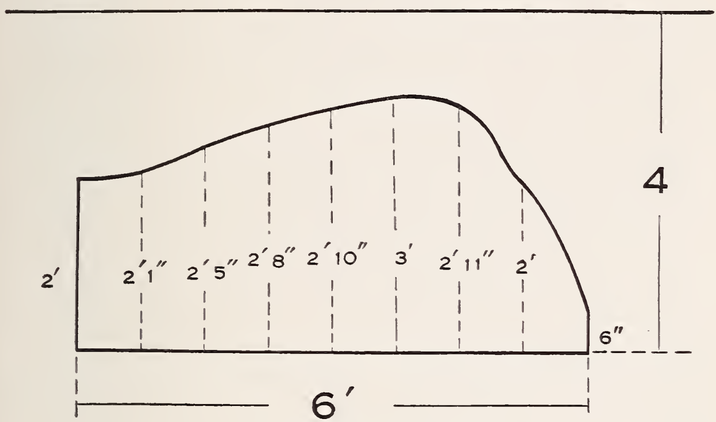


FIG. 169.

281. Find x so that the force on P equals that on Q (Fig. 167).

282. Find the force on this vertically submerged trapezoid (Fig. 168).

283 A solid rectangular masonry dam weighing 140 pounds per cubic foot is 12′ deep. Find the width, if the dam is to be 60 per cent wider than the width at which it would just overturn under a head of 10′ on one face.

284. Find magnitude and point of application of the force on the area in Fig. 169 if submerged vertically in fresh water.

285. What depth of water in a cylindrical jar of radius r produces the same force on the sides as on the bottom?

CHAPTER VII

DEFORMABLE BODIES

83. Introduction.—The members of any machine or structure are acted upon by forces and transmit forces. In Chapter IV we studied the forces acting upon and transmitted by the various members of a frame, and we can study with the aid of kinetics the forces acting upon and transmitted by the moving members of a machine. In both cases we assume that the members are rigid bodies, that is, remain unaltered regardless of the magnitude of the forces acting upon them. Experience, however, tells us that actually there is a limit to the magnitude of the forces which can be applied to any member with safety, and that this limit is related to the shape, dimensions and material of the member. In this chapter we consider the behavior of material under the action of force. This study is sometimes called "Strength of Material." In treating such problems it is necessary to use results obtained by experiment with actual materials. Tables giving these results may be found in any handbook and a brief table follows:

Material	Wt. lbs. per cu. ft.	Ultimate strength in lbs. per sq. in.			Elastic limit in lbs. per sq. in.	Modulus in lbs. per sq. in.	Factors of safety	
		Tension	Compression	Shear			Steady load	Shocks
Struct. steel	490	60000	60000	50000	35000	30×10^6	4 to 10	
Cast iron	450	22500	90000	20000	6000	15×10^6	8 to 20	
Wrought iron	480	50000	40000	40000	25000	28×10^6	4 to 10	
Yellow pine	40	9000	7000	1500	3000	150×10^3	8 to 15	
Oak	48	10000	6000	4000	3000	150×10^3	8 to 15	
Brick	125	500	3000	1000	.....	....	15 to 40	
Concrete	150	300	2500	1000	1000	2000×10^3	15 to 40	

84. Normal Stress.—A rod acted upon by two equal and opposite forces P (Fig. 170), applied to its ends in the direction of

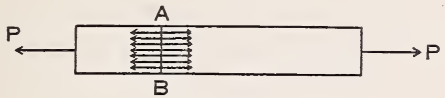


FIG. 170.

its axis, is said to be subject to a "Normal Stress" and the force P is called the "Load." If we consider a cross section of the bar by a plane perpendicular to its axis, equal and opposite forces, P, must act on either side of the section to preserve equilibrium. The force P is assumed to be distributed uniformly over the cross-sectional area. The load per unit cross-sectional area is called the "Stress" and we shall denote it by p. Thus,

$$\text{Stress} = \text{Load} \div \text{Cross Section}.$$

If A denote the cross section, $p = P/A$.

The stress is called "Tensile Stress" and the bar is under tension when the two forces P tend to separate the parts on the two sides of the section, and is called "Compressive Stress," and the bar is under compression when the forces tend to push the parts together. Tensile stresses and compressive stresses are called "Normal Stresses" since they act at right angles to the section of the bar.

Example.—A hollow cast-iron column of six inches exterior and five inches interior diameter sustains a load of 50,000 pounds. Find the compressive stress due to this load.

Solution.—

$\pi(36-25)/4 =$ area of cross section of metal in square inches.

$$P = 50{,}000 \text{ lbs.}$$

$$\therefore p = P/A = 50{,}000 \div 11\pi/4 = 50{,}000 \cdot \tfrac{2}{11} \cdot \tfrac{7}{11}$$

$$= 5{,}800 \text{ lbs. per sq. in. (Approx)}$$

Problems

286. A wrought-iron rod one inch in diameter is under a tension of 8000 pounds. Find the stress.

287. The diameter of the piston head of a steam engine is 16 inches and that of the piston rod $2\frac{1}{2}$ inches. If the engine is working under a steam pressure of 100 pounds per square inch, what is the maximum stress in the rod?

288. The head of an engine cylinder is 10 inches in diameter and the steam pressure applied to it is 125 pounds per square inch. The head is fastened on by 8 steel bolts, $\frac{1}{2}$ inch in diameter. Find the tensile stress in each bolt.

289. Calculate the size of a square wrought-iron bar to stand a pull of 3000 pounds without breaking.

85. Longitudinal Strain.—If a bar is stressed in the way de-

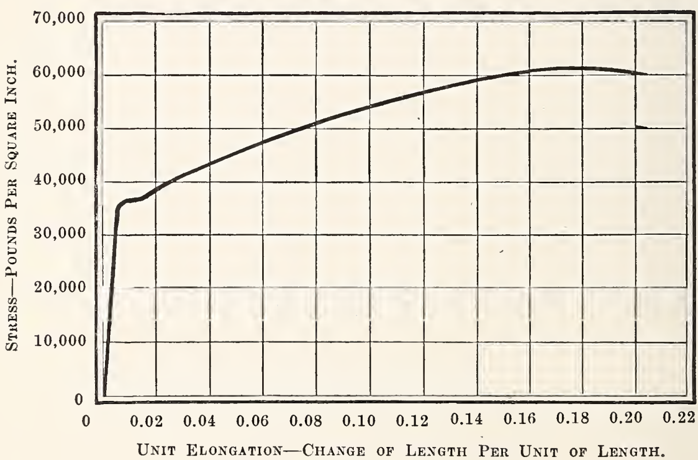


FIG. 171.—Stress-Strain Diagram.

scribed above a change in length takes place. It is usual to consider the change in length per unit length and to call this the

"strain," which we shall denote by e. Thus, if a bar l inches long is elongated or contracted x inches, $e = x/l$.

86. Stress-Strain Diagram.—If a test piece of material is placed in a testing machine and the load slowly applied, we find that the deformation of the piece is gradual and proportional to the stress until the stress reaches a certain value. This value is called the "Elastic Limit," since, if the stress has not exceeded it, the piece will resume approximately its original shape when the load is removed. As the stress increases beyond the elastic limit the deformation increases more rapidly until the stress reaches a value when again there is a marked difference in the way the deformation changes. This second value of the stress is called the "Yield Point." From the yield point on, the stretch is very rapid with little change of stress until finally the piece breaks. The value of the stress when the rupture takes place is called the "Ultimate Strength."

As might be supposed, materials vary widely in their ability to withstand different kinds of loads. Steel is about equally strong in tension and compression. On the other hand, concrete, while strong in compression, can stand little tension and hence must be reinforced when it is to carry a tensile load.

87. Hooke's Law—Modulus of Elasticity.—Hooke's Law states that within the elastic limit the ratio of stress to strain is constant. This ratio is called the "Modulus of Elasticity" or "Young's Modulus" and is represented by E. Thus, $E = p/e$. As stated in the preceding paragraph, the variables p and e are connected by a straight-line law. Since e is merely the ratio of two lengths it is independent of the units of measurement, and hence E is always measured in the same units as p, usually pounds and square inches. The value of E varies for different kinds of materials but not with the shape or size of the piece.

Example.—A bar of steel, 10 feet long and 3 square inches in section, stretches $\frac{2}{25}$ inches under a load of 60,000 pounds. Find E.

The stress, $p = 60{,}000/3$ lbs. per sq. in.
The strain, $e = \frac{2}{25} \div 120$.
The modulus, $E = p/e = 30{,}000{,}000$ lbs. per sq. in.

Problems

290. How much is a 100-foot steel tape, $\frac{3}{8}$ inch wide and $\frac{1}{40}$ inch thick, lengthened by a pull of 40 pounds?

291. A wrought-iron tie rod, $1\frac{1}{4}$ inches in diameter, is stretched $\frac{1}{2}$ inch by a pull of 18,000 pounds. Find the original length of the rod.

292. If the breaking strength of a wire rope is 90,000 pounds per square inch, find the greatest load which an elevator rope $\frac{1}{2}$ inch in diameter can sustain.

293. If a steel shaft is suspended vertically from one end, how long will it have to be in order to break under its own weight?

294. Find the modulus of elasticity of a copper wire 26 feet long and 0.09 inches in diameter if it stretches 0.4 inches under a pull of 120 pounds.

88. Working Stress and Factor of Safety.—In engineering practice the allowable stress in any material must be well within the elastic limit, for beyond that point the stress causes a permanent set in the material and it will not even approximately assume its original shape when relieved of the load. This allowable stress is called the "Working Stress" and is the stress on which the design of the member depends. It is obtained by dividing the ultimate strength by a constant called the "Factor of Safety." The choice of this factor depends on many things, among which are the uniformity of the material, whether the loads are gradually or suddenly applied, etc. The choice of the proper factor is largely a matter of experience, and, for various materials and various kinds of loads, it runs from 4 to 35 or 40.

Problems

295. Find the factor of safety in Probs. 286, 287, 288.

296. A short wooden (oak) column, 10 inches in diameter, sustains a load of 9500 pounds. Find the factor of safety. How great a load could this column sustain if the factor of safety were 8?

89. Shear.—We have seen that the normal stress acts at right angles to the section of the bar and tends to move the molecules in that direction. It often happens that the member is so loaded that the molecules tend to move past each other in a direction parallel to the section. Such an action is called a "Shear" and the load a "Shearing Force." The shearing force per unit of area is called "Shearing" or "Tangential Stress," is denoted by q, and is measured in pounds per square inch. Thus, if F is the shearing force and A the cross-sectional area, $q=F/A$.

Two plates bearing a longitudinal load and held together by a rivet present a good illustration of shearing stress on the rivet (Fig. 172).

If the rivet fits the holes closely and the load is sufficiently

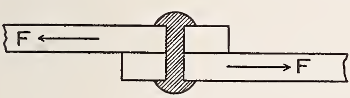


FIG. 172.

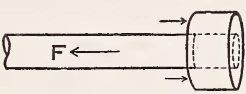


FIG. 173.

great, it will be sheared off smoothly as though a plane had been passed perpendicular to its axis.

If a bolt is in tension a shearing force acts on its head, tending to strip it from the bolt (Fig. 173). The shearing stress here acts along the cylindrical surface which is the continuation of the surface of the shank.

Example 1.—A steel bolt 1 inch in diameter with a head $1\frac{1}{4}$ inches deep is subjected to a tension of 15,000 pounds. Let it be required to find the shearing stress in the head.

The area over which the shear acts is the lateral surface of a right circular cylinder which equals $\pi \cdot 1 \cdot 5/4 = 3.93$ square inches. Then $q = 15{,}000/3.93 = 3{,}817$ pounds per square inch. Taking the ultimate strength in shear to be 50,000 pounds per square inch, the factor of safety against shearing is $50{,}000/3{,}817 = 13.1$.

Example 2.—A boiler, 48 inches in diameter, is made of $\frac{1}{2}$ inch plate joined by a single-riveted lap joint. The rivets are $\frac{3}{4}$ inches in diameter and the pitch is 2 inches. If the internal steam pressure is 120 pounds per square inch, what will be the shearing stress in the rivets?

A single-riveted lap joint is formed by overlapping plates held in position by a single row of rivets. The pitch is the distance between the centers of the rivets.

From Art. 79 the force on each rivet is $P = p'rh$, where p' is the steam pressure, r the radius of the boiler in inches and h is the pitch.

Thus,

$$P = 120 \cdot 24 \cdot 2 = 5760 \text{ lbs. per sq. in.}$$

Then

$$q = P/A = 5760 \cdot \frac{64}{9\pi} = 13{,}032 \text{ lbs. per sq. in.}$$

If there had been two rows of rivets, called a double-riveted lap joint, the stress in each rivet would be half what it was in the first case. A more complete discussion of riveted joints can be found in a handbook.

Problems

297. Compute the shearing stress in the heads of the bolts in Problem 288, if the heads are $\frac{3}{4}$ inches deep. Find also the factor of safety against shearing.

298. Compute the shearing stress on the wrought-iron rivets of a boiler 3 feet in diameter, which is carrying a steam pressure of 120 pounds per square inch, if the joint is a double-riveted lap joint with $\frac{1}{2}$ inch rivets and $1\frac{1}{4}$ inch pitch. Find factor of safety against shear.

90. Equal Strength.—Suppose it is required to find the thickness of the head of a steel bolt 1 inch in diameter so that the bolt will be equally strong in tension and shear.

The tensile load that the bolt will carry will be $\pi \cdot 1^2 \cdot 60{,}000/4$. The total shear (see Fig. 173) which the head will carry will be $\pi \cdot 1 \cdot t \cdot 50{,}000$ where t is the thickness of the head. Since these loads must be equal

$$\pi \cdot 1^2 \cdot 60{,}000/4 = \pi \cdot 1 \cdot t \cdot 50{,}000.$$

Therefore,

$$t = \frac{60000}{4 \cdot 50000} = 3/10 \text{ inches.}$$

If a riveted joint, as that of a boiler, fails because the plate ruptures, the break will occur along the outer row of rivets, since the force required to produce the break is least here. Rivets lessen the strength of the plate, since metal is removed for the holes. Hence the strength of the plate must be taken at its weakest section, *i. e.*, along a row of rivets.

Example.—A boiler plate is $\frac{1}{2}$ inch thick and there is a double-riveted lap joint of $\frac{3}{4}$ inch rivets; what must be the pitch of the rivets for equal strength of the joint in tension and in shear?

Since the material of the rivets and plate is the same, the same factor of safety would be used for tension and shear, and in the numerical work would cancel out. Hence instead of the working stresses the ultimate strengths in tension and shear, 60,000 pounds and 50,000 pounds per square inch, respectively, can be used.

The area of a section of the plate between two consecutive rivet holes will be $\frac{1}{2}(p-\frac{3}{4})$ where p is the pitch of the rivets. The strength of the plate is then

$$60{,}000 \cdot \tfrac{1}{2} \cdot (p-\tfrac{3}{4}).$$

Since there are two rows of rivets, the total load will be borne by two rivets whose strength in shear will be

$$2 \cdot 50{,}000 \cdot \pi \cdot 9/(4 \cdot 16),$$
$$60{,}000 \cdot \tfrac{1}{2} \cdot (p - \tfrac{3}{4}) = 2 \cdot 50{,}000 \cdot \pi \cdot 9/(4 \cdot 16),$$

whence

$$p = 2.22 \text{ inches.}$$

Problems

299. If the boiler in Prob. 298 should explode, would the failure be due to the shearing of the rivets or to the rupture of the steel plate (assume the plate $\frac{5}{16}$ inch thick)?

300. The efficiency of a joint is its minimum strength divided by the strength of the unpunched plate. Find the efficiency of the joints in Prob. 298.

301. If the thickness of a boiler is t, the diameter of the rivets d and the pitch of the rivets p, find the relation between d and t that will make the joint equally strong in tension and in shear, assuming a double-riveted lap joint.

302. A bolt $1\frac{1}{2}''$ in diameter has a head $\frac{3}{4}''$ thick which shears off under a load of 12,000 pounds. Find the ultimate shearing strength of the bolt material.

303. Two $\frac{1}{2}''$ plates are double-riveted with single butt-strap joint, the rivets being $\frac{3}{4}''$ in diameter. If $q = \frac{4}{5}p$, find the pitch for equal strength.

304. A single-riveted lap joint, plate $\frac{3}{4}''$ thick, $8''$ wide, contains 3 rivets, each $\frac{3}{4}''$ in diameter. If the plate is subjected to a pull of 12,000 pounds, find the tensile stress of the plate, the shearing stress on the rivets, and the efficiency of the joint.

305. Determine the thickness of head for $1\frac{1}{4}''$ wrought-iron bolt, if the tensile strength of the bolt is to be equal to the strength of the head against shearing.

91. Beams.—If a horizontal bar is under the action of forces perpendicular to its axis, it is called a beam. The forces which act upon the beam may be either loads concentrated at points or distributed over the beam, this distribution being uniform or varying in any manner. Beams are classified according to the manner in which they are supported. A beam may simply rest

on any one of its supports, in which case it is said to be *supported* at this point; or it may be held rigidly fixed at the support, in which case it is said to be *fixed* at the point. A beam supported at its ends (Fig. a) is called a *simple* beam; a beam fixed at both

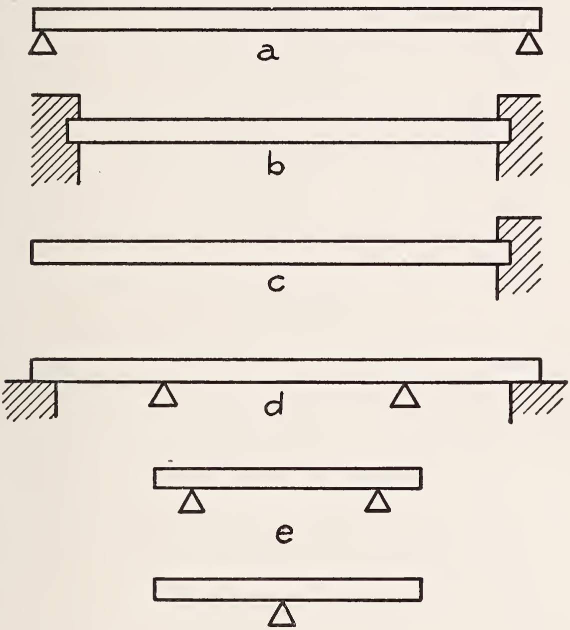


FIG. 174.

ends is called a *built-in* beam (Fig. b); a beam free, *i. e.*, not supported at all, at one end and fixed at the other is called a *cantilever* beam (Fig. c); a beam extending over more than two supports is called a *continuous* beam (Fig. d); a beam overhanging its supports is called an *overhanging* beam (Fig. e).

92. Kinds of Stresses in Beams.—If a cantilever beam is loaded at its outer end, the load tends to rupture the beam by *bending* as shown in Fig. 175.

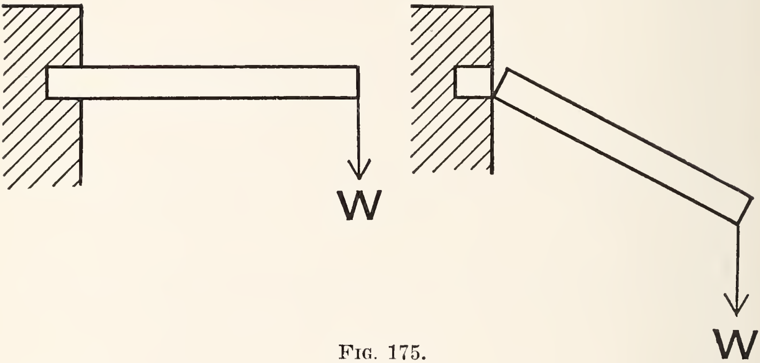


FIG. 175.

If, on the other hand, a concentrated load is applied close to one of the supports of a simple beam, the load tends to rupture the beam by *shearing;* that is, to make the parts of the beam

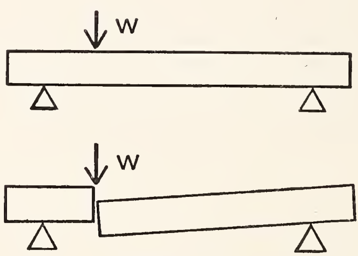


FIG. 176.

separate by slipping past each other in a vertical plane (Fig. 176). These are the two principal kinds of stresses, or tendencies to rupture, occurring in beams. In general at any point of

a beam both kinds are found. In order that the beam may support the required load it must be strong enough to resist both of them. The following sections will deal with the determination of the magnitude of these stresses due to any kind of loading.

93. Shearing Force.—If a beam with any kind of loading and supporting forces, for example those in Fig. 177, is thought of as

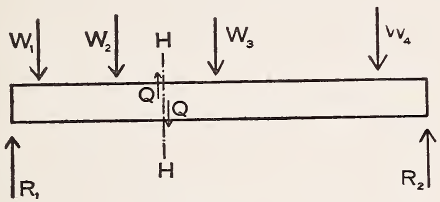


FIG. 177.

cut at any point by a plane HH, each part of the beam must be in equilibrium under the action of all the forces which act on it. Now just as in applying the method of sections to frames (Art. 51) these forces are of two kinds, the external forces and those due to the stress at the plane of section, *i. e.*, the force exerted by one part of the beam on the other. (Note that the word "stress" as used in this chapter refers always to *intensity* of stress, not to total force as in Chap. IV.) Thus the vertical forces acting on the left-hand part of the beam in the figure are the two loads W_1 and W_2, the supporting force R_1, and Q, the shearing force at the section HH. As all of these but the last are known or readily found, we can find the last also by applying one of the conditions for equilibrium, namely, by setting $\Sigma Y = 0$. Thus *the shearing force at any section is numerically equal to the algebraic sum of all the vertical forces acting on one side of the section.*

94. Algebraic Sign of the Shearing Force.—As the shearing force always tends to move one part of the beam up and the other down, it is necessary to establish a definite convention for its algebraic sign. The shearing force is called positive at any section if the tendency of the external forces is to move the part to the left of the section up and that to the right down, negative in the contrary case.

The student should note carefully that generally the shearing force varies with the position of the section and will be constant for all sections of the beam only in one special case.

95. Shearing-Force Diagrams.—It is often necessary to find the shearing force for every section of a beam. To facilitate this calculation we plot the value of the shearing force as an ordinate vertically above the section for which it has been calculated. The locus of these points is a curve or diagram. The origin is taken at the left end of the beam, and the units on the horizontal axis represent units of length, while those on the vertical axis represent units of force.

Example 1.—Draw the S. F. diagram for a simple beam, 12

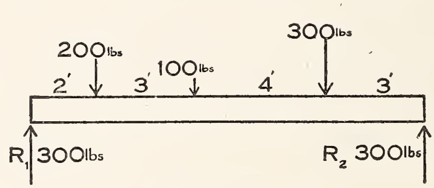


FIG. 178.

feet long, loaded as shown in Fig. 178, the weight of the beam being neglected.

The reactions R_1 and R_2 are found by taking moments about the supports and using as a check that $R_1+R_2=$sum of the loads. We start at the left reaction and lay off $R_1=300$ to scale, using any convenient unit (Fig. 179). Since there are no forces

between R_1 and the 200-pound load, the S. F. must have the same value throughout this interval and the diagram is the line *ab*. When we pass over the 200-pound load the S. F. drops to 100 pounds, since the resultant force to the left of this section is $300-200=100$ pounds. The diagram falls to *c* and continues horizontal until the 100-pound load is reached at *d*, and then drops to *e*. The S. F. at this section and any section beyond up to the 300-pound load is zero, and the curve *ef* lies along the beam. From *f* the diagram drops to *g* (-300) by passing over the load of 300 pounds. It continues in the horizontal direction to *h*, where the value is -300, or the negative of the reaction R_2.

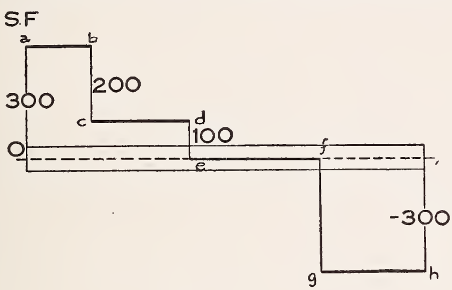


FIG. 179.

The S. F. at the supports is always numerically equal to the reactions. The complete S. F. diagram then is the broken line *abcdefgh*. If we denote the shearing force at any section x units from the left end by $Q(x)$, we may write the equations for the S. F. curve as follows:

$Q(x)=300$ for x between 0 and 2 feet; $Q(x)=100$ for x between 2 feet and 5 feet; $Q(x)=0$ for x between 5 feet and 9 feet; $Q(x)=-300$ for x between 9 feet and 12 feet.

Example 2.—Draw the S. F. diagram for a simple beam, of length l feet, uniformly loaded with w pounds per foot run.

The total load will be wl pounds and as the loading is symmetrical the reactions are each $wl/2$ pounds. Now if we take any section AB, at a distance x from the left support, the S. F. will be $wl/2-wx$. Thus the equation of the curve is

$$Q(x)=wl/2-wx.$$

This is the equation of a straight line and the diagram is the line *acb* shown in Fig. 180. The S. F. is zero when $x=\frac{1}{2}l$.

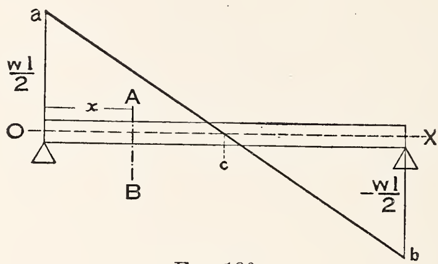


FIG. 180.

Problems

306. Construct the S. F. diagram for a cantilever beam of length 15 feet with a uniformly distributed load of 250 pounds. What is the S. F. 3 feet from the fixed end? 10 feet from the fixed end?

307. Draw the S. F. diagram for a simple beam 20 feet long with a concentrated load of 1000 pounds at the center and another of 540 pounds 3 feet from the right end. What is the S. F. 2 feet from the left end? 12 feet from the left end? 1 foot from the right end?

308. Draw the S. F. diagram for a cantilever beam of length 12 feet with a uniform load of 40 pounds per foot run and a concentrated load of 2000 pounds 3 feet from the free end.

309. A simple beam 16 feet long carries a uniform load of 40 pounds per linear foot, and two concentrated loads, 240 pounds 3 feet from the left end and 180 pounds 4 feet from the right end. Draw the S. F. diagram and find the S. F. 4 feet from each end.

310. A simple beam 20 feet long carries 2000 pounds 5 feet from the left end, 5000 pounds 4 feet from the right end, and in addition a load of 200 pounds per foot run; find the S. F. at the ends and center and draw the S. F. diagram.

311. A beam 20 feet long, supported at its ends, carries 8 tons 6 feet from the left end, 3 tons 9 feet from the left end, and 2

tons 14 feet from the left end. Draw the S. F. curve and give value of S. F. at the center.

312. A beam 10 feet long has its left end fixed in a wall, and it carries a uniform load of 80 pounds per foot run, together with a concentrated load of 600 pounds 7 feet from the wall. Draw the S. F. curve and give the value of the S. F. $8\frac{1}{2}$ feet from the wall.

96. Bending Moment.—In considering the shearing force in a beam we applied only one condition of equilibrium ($\Sigma y=0$) to the part of the beam on one side of the section. From the definition of a beam, the external loads have no horizontal com-

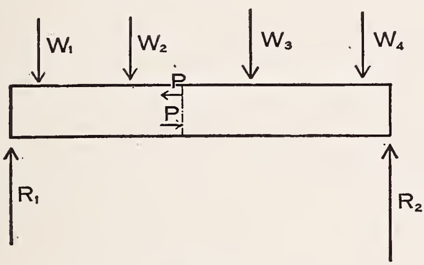


FIG. 181.

ponent. So the horizontal components of the forces due to the internal stresses must amount to zero at any section (Fig. 181). It follows from this that at any section where there is normal stress across the section, it must be tensile in one part of the plane of section and compressive in the other. That is to say, these forces form a couple.

Before considering the exact distribution over the section of these normal forces, we can determine the magnitude of the couple they form by applying the condition of equilibrium $\Sigma M=0$ to the part of the beam on one side of the section and taking moments about an axis in the section in order to eliminate the shearing force at the section.

Example.—A simple beam 10 feet long is loaded with weights of 3000, 2000, and 6000 pounds at points 2, 4, and 8 feet respectively from the left end. Find the magnitude of the bending moment at a point 6 feet from the left end.

The supporting forces 4800 and 6200 are found by the ordinary method of statics applied to the beam as a whole (Fig. 182). Considering now the part to the left of the section *HH*, the moments of the external forces about a point in *HH* will be $4800 \cdot 6$ or 28800 clockwise and $3000 \cdot 4 + 2000 \cdot 2 = 16000$ counter-clockwise. That is, the external forces exert a total moment of

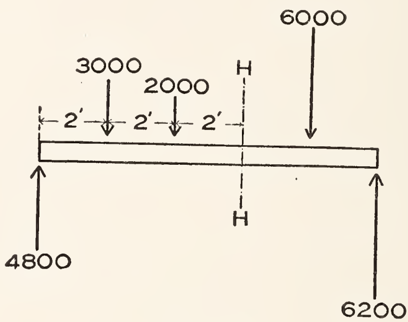


FIG. 182.

$28800 - 16000$ or 12800 pound-feet clockwise. But since the left-hand part of the beam is in equilibrium the internal forces at the section must furnish a moment of 12800 pound-feet counter-clockwise.

The principle exemplified here is sometimes expressed in the statement that the *bending moment is equal to the resisting moment, the bending moment being the algebraic sum of the moments of the external forces on the part of the beam on one side of the section,* the resisting moment being that of the force exerted on this part of the beam by the other part, and transmitted by the internal stresses at the section.

The student should notice that, like the shearing force, the bending moment will vary from point to point in the beam.

97. Sign of the Bending Moment.—As with the shearing force, a definite convention for the sign of the bending moment is also necessary, since it is obviously clockwise on one side of the section and counter-clockwise on the other. The bending moment is called *positive* at any point when the *external* forces tend to turn *clockwise* the part of the beam to the *left* of the section, and of course negative in the contrary case. The reason for this convention is that when the moments act in the way we call positive they tend to bend the beam into a curve which is concave upward, that is, has positive curvature. It may be worth noting that if we regard it from this point of view we can see that the bending moment retains its sign even if we look at the beam from the opposite side, thus interchanging left and right. This is not the case with the sign of the shearing force.

Example.—A simple beam 12 feet long has a uniformly distributed load of 300 pounds per foot run on the left half. Find the bending moments at 4 feet and 8 feet from the left end.

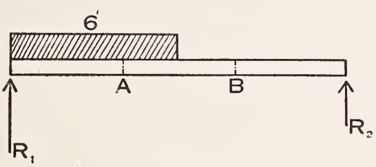


FIG. 183.

Solution.—To find the supporting forces we may regard the load as concentrated at its center of gravity. This gives $R_1=1350$ pounds and $R_2=450$ pounds (Fig. 183).

The bending moment at A, the first point, is $4R_1$ clockwise and the moment (counter-clockwise) of the part of the load to the left of A. This latter is $4\cdot300$ or 1200 pounds acting (on an average) at a distance of 2 feet from A. Hence the bending moment at A is $4R_1-2\cdot1200=5400-2400=3000$ pound-feet. We may find the bending moment at B in the same way; that is,

$$8\cdot1350-5\cdot1800=10800-9000=1800 \text{ lb.-ft.}$$

It is easier, however, to find it as the total *counter-clockwise* moment to the *right* of the section, *i. e.*, $4R_2 = 4 \cdot 450 = 1800$ pound-feet.

98. Bending-Moment Diagram.—Since the bending moment varies from point to point as we pass from one end of the beam to the other, the easiest way to get the general idea of its behavior is to draw its graph regarded as a function of x, the distance from one end of the beam.

Example 1.—A simple beam 10 feet long carries loads of 800 and 600 pounds at distances of 2 feet and 6 feet respectively from the left end. Draw the bending-moment diagram.

Solution.—The supporting forces are 880 and 520 pounds at the left and right ends respectively.

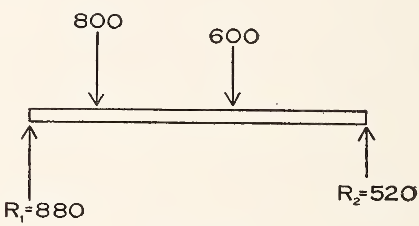


FIG. 184.

The length of the beam is divided into three intervals by the two concentrated loads. In the first of these if we take the bending moment at any point distant x feet from the left end, the only external force to the left is the reaction R_1. The bending moment M_1 then is $880x$. We must be careful to note that this formula holds *only* until we reach the first load. After that the moment of this load must be brought in, giving

$$M_2 = 880x - 800(x-2) = 80x + 1600.$$

Similarly in the third interval,

$$M_3 = 880x - 800(x-2) - 600(x-6),$$
$$= -520x + 5200.$$

We have a check here, as $M_3=0$ when $x=10$, which we can easily see is correct by considering the right-hand part. The

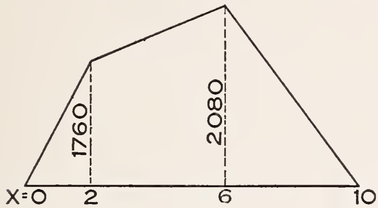


FIG. 185.

graph (Fig. 185) then is determined as follows:

$$M_1=880x \qquad 0 \leqq x \leqq 2 \qquad (1)$$
$$M_2=80x+1600 \qquad 2 \leqq x \leqq 6 \qquad (2)$$
$$M_3=-520x+5200 \qquad 6 \leqq x \leqq 10 \qquad (3)$$

It is important for the student to note that a formula which holds for one interval is entirely incorrect in any other. Two consecutive formulas, however, agree at their common end point. Thus (1) and (2) both give 1760 when $x=2$ and (2) and (3) both give 2080 when $x=6$

In this problem each of the formulas for M is the first degree in x, hence the graph in each interval is a straight line. It is easy to see that this will always be so for concentrated loads since the distance x enters only in the arm of the moment. Hence in order to draw the B. M. curve we need compute the B. M. only at the points where the loads are applied, plot these and connect them by straight lines.

Example 2.—Draw the B. M. curve for the example of Art. 97 and find the maximum bending moment in the beam.

Solution.—In the first interval,

$$0 \leqq x \leqq 6, \ M_1=1350x-x/2 \cdot (300x),$$
$$=1350x-150x^2.$$

For the rest of the beam

$$6 \leqq x \leqq 12, \ M_2=1350x-1800(x-3)=-450x+5400.$$

To find the maximum value of M_1, we set the derivative of $1350x-150x^2$ equal to zero, $1350-300x=0$ or $x=4\frac{1}{2}$. There is no maximum in the second interval, as M is there always de-

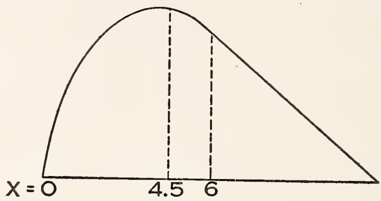


FIG. 186.

creasing. When $x=4\frac{1}{2}$, $M=3037\frac{1}{2}$, the maximum B. M. (Fig. 186).

Problems

313. A simple beam 20 feet long has loads of 300, 450 and 250 pounds placed at distances of 5, 11, and 17 feet respectively from the left end. Find the B. M. and S. F. at 6, 10, and 15 feet from the left end.

314. A simple beam 29 feet long carries a load of 80 pounds per foot run. Find the B. M. and S. F. at 10, 19, and 26 feet from the left end.

315. If the beam in Prob. 314 also carries a concentrated load of 1000 pounds at its midpoint, find the B. M. and S. F. for the same points as in Prob. 314.

316. Draw the B. M. and S. F. diagrams for the problems in 307, 308, and 309.

317. A beam 8 feet long is fixed at the left end and carries a concentrated load of 300 pounds 3 feet from the right end. Draw the B. M. and S. F. diagrams.

318. A beam, 12 feet long, supported at its ends, carries concentrated loads of 300 and 700 pounds at points 4 and 9 feet respectively from the left end. Draw the B. M. and S. F. diagrams.

319. A 12-foot beam, supported at the ends, carries 6 tons 8 feet from the left end, and a continuous load of $\frac{2}{3}$ ton per foot-run. Draw the S. F. and B. M. curves and find the B. M. 10 feet from the left end.

99. Relations Connecting Loading, Shearing Force and Bending Moment.—We have seen in the examples of the preceding section from the diagrams of Q and M that where there is a concentrated load there is a discontinuity in the value of Q. In M itself there is no discontinuity, but there is in its slope. Also there is a discontinuity in the slope of Q, where there is a sudden change in the rate of loading in the case of a distributed load. At other points, however, we found that these graphs were straight lines or smooth curves. We shall now derive two very simple relations connecting M, Q, and L, the rate of loading in a distributed load.

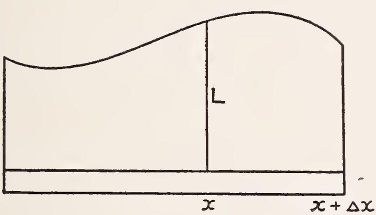


FIG. 187.

Let Q and $Q+\Delta Q$ be the values of the shearing force at two points distant x and $x+\Delta x$ from the left end of the beam (Fig. 187). We have seen (Art. 93) that Q is equal to the algebraic sum of the vertical forces to the left of x, likewise $Q+\Delta Q$ is the sum of those to the left of $x+\Delta x$. Then ΔQ must be merely the additional force exerted between x and $x+\Delta x$. If there is no concentrated load in this interval, this force will be Δx multiplied by $\overline{L}$, the average value of L in this interval.

We have, then, $\Delta Q/\Delta x=\overline{L}$; or, since this will approach L as Δx approaches zero,

$$\frac{dQ}{dx}=L, \tag{1}$$

where L is the *rate* of loading, *i. e.*, the load *per unit of length* at the point desired. This of course does not apply at a point where there is a concentrated load; but the same discussion covers this case also. For if in the interval Δx there is a concentrated load W, ΔQ will then be $W+L\Delta x$; that is, in addition to the continuous change in Q due to the distributed load of intensity L there is a sudden change of magnitude W. If we write equation (1) in the integral form,

$$Q=\int_0^x L\,dx,$$

we can include both kinds of loading if we understand the integral in a broad sense to include the addition of concentrated loads. The rate of change in the shearing force is equal to the continuous load per unit of length and any concentrated load produces a sudden change in the shearing force of an equal magnitude and direction.

To obtain the relation between M and Q, consider the resultant of all vertical forces which act to the left of a point distant x from the left end. This (by Art. 93) is equal in magnitude and direction to the shearing force Q at this point. Let its point of application be at a distance $\bar{x}$ from the left end. Then its moment about x is

$$M=(x-\bar{x})Q$$

and this is the bending moment at the point x. If we denote by $M+\Delta M$ the bending moment at a new point $x+\Delta x$, this will consist of two parts, the original force acting at $\bar{x}$ and the additional force ΔQ acting in the interval Δx. The arm of the original force will now be $x+\Delta x-\bar{x}$ and that of the new force ΔQ will be some fraction of Δx, $\theta\Delta x$, $(0\leqq\theta\leqq1)$.

The new moment will then be

$$M+\Delta M=(x+\Delta x-\bar{x})Q+\theta\Delta x\cdot\Delta Q.$$

Hence

$$\Delta M = Q\Delta x + \theta\Delta x\Delta Q,$$

and

$$\frac{\Delta M}{\Delta x} = Q + \theta\Delta Q.$$

For a continuous loading ΔQ approaches zero with Δx and hence

$$\frac{dM}{dx} = Q,$$

or

$$M = \int Q\,dx.$$

100. As in the case of the shearing force, the integral proper takes account only of the case where the load is continuous. It is usual to apply this integral separately to each interval in which there is no discontinuity. Its use and that of the integral expression for the shear are best explained by an example.

Example 1.—A simple beam 20 feet long carries a variable load on the left half whose intensity is proportional to the square of the distance from the left end and varies from zero at the end to 300 pounds per foot at the middle. On the right half it carries a uniform load of 300 pounds per foot and in addition a concentrated load of 4000 pounds at a point 2 feet to the right of the middle. Find the general expressions for the shearing force and bending moment at all points of the beam.

Solution.—The intensity of loading is of the form kx^2 on the left half. Since this equals 300 when $x=10$, we have $k=3$. If R_1 and R_2 are the supporting forces,

$$R_1 + R_2 = \int_0^{10} 3x^2\,dx + 10\times 300 + 4000 = 8000.$$

Taking moments about the left end,

$$20R_2 = \int_0^{10} 3x^3\,dx + 10\times 300\times 15 + 4000\times 12$$
$$= 7500 + 45000 + 48000 = 100500.$$
$$\therefore R_2 = 5025.$$

Hence

$$R_1 = 8000 - 5025 = 2975.$$

In the first interval, $0 \leqq x \leqq 10$, we have $L = -3x^2$, negative because the loading is a downward force.

$$Q = \int -3x^2\, dx = -x^3 + C.$$

Now when $x = 0$, $Q = R_1 = 2975$, since R_1 is then the only force on the left. Hence

$$C = 2975,$$

and

$$Q = 2975 - x^3. \qquad (1)$$

Integrating again,

$$M = 2975x - \tfrac{1}{4}x^4 + C_1.$$

Since the beam is freely supported, $M = 0$ at the ends. Hence $C_1 = 0$ and

$$M = 2975x - \tfrac{1}{4}x^4. \qquad (2)$$

Before proceeding further let the student note that the equations (1) and (2) hold only in the first interval, *i. e.*, when $0 \leqq x \leqq 10$.

In the second interval, $10 \leqq x \leqq 12$, $L = -300$.

$$\therefore Q = -300x + C_2.$$

C_2 must have a value that will make $Q(10)$ have the same value as that given by (1), namely 1975.

$$\therefore Q(10) = 1975 = -3000 + C_2,\ C_2 = 4975,$$

and

$$Q = 4975 - 300\,x. \qquad (3)$$

Integrating again and determining the constant for M in the same way,

$$M = 4975x - 150x^2 - 7500. \qquad (4)$$

In the third interval, $12 \leqq x \leqq 20$, we have the same distributed load, but a change in the equations because of passing over the concentrated load of 4000. This causes a sudden drop in Q, so that in this interval

$$Q = 975 - 300x. \qquad (5)$$

Integrating this and determining the constant so that $M(12)$ will agree with equation (4),

$$M = 975x - 150x^2 + 40500. \qquad (6)$$

We may check the correctness of the work by observing in (5) and (6) that $Q(20)=-5025=-R_2$ and $M(20)=0$.

The shearing force is given by equations (1), (3), and (5), each applying in its own interval, and the bending moment similarly by equations (2), (4), and (6).

Example 2.—Draw the S. F. and B. M. diagrams, for the beam in the preceding example.

Solution.—The curve for Q in the first interval, $0 \leqq x \leqq 10$, equation (1), starts with $Q=2975$ and the slope zero, and ends with $Q=1975$ and the slope -300, where $x=10$. Since x is measured in feet, Q in pounds, and M in pound-feet, there is no connection between the scales by which they are represented. So

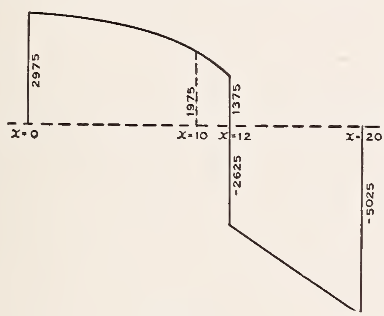


FIG. 188.

each may be chosen at convenience independently of the others. When it is said that the slope of the Q curve is -300 when $x=10$, this merely means that the curve is descending at the rate of 300 units on the pound scale for each unit advance on the foot scale. In each of the other two intervals Q is of the first degree and so the graph is a straight line, which is easily drawn as soon as we have two points on it (Fig. 188). The graph of M may be drawn directly from the equations (2), (4), and (6), but we should note that its slope should always be equal to the height of the Q curve for the same value of x. From this we can see that no part of the graph of M is a straight line; that M increases from $x=0$ to $x=12$ while Q is positive, whereas it decreases after that because Q is negative. This at once tells us

that the greatest bending moment occurs when $x=12$, and is equal to 30,600 pound-feet (Fig. 189). Note that at this point the graph of Q crosses the axis.

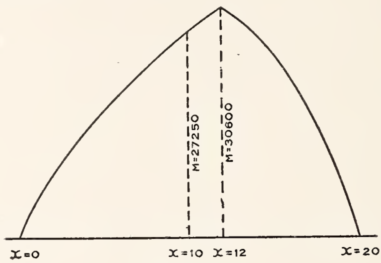


FIG. 189.

101. Dangerous Section.—In designing a beam it is necessary to make it strong enough to resist the greatest bending moment that may be applied to it, and this of course means the greatest *numerically,* regardless of sign, since the sign tells us the direction of bending. The point where this numerically greatest bending moment occurs is called the *dangerous section.* Thus, in the example solved just above, the dangerous section occurs where $x=12$ feet, since there the value of M (30,600 pound-feet) was found to be greater than at any other section of the beam. In finding the dangerous section, it is important for the student to realize the necessity of having a clear idea of the general character of the bending-moment curve rather than applying any mechanical rule.

Example 1.—A cantilever beam 6 feet long bears a concentrated load of 1500 pounds at its outer end. Draw S. F. and B. M. diagrams, and determine the dangerous section and greatest bending moment.

Solution.—Let us take the origin at the wall end. The supporting force must be equal to the load 1500, and of course acts upward; hence the S. F. is initially 1500, and since there is no

distributed load it is constant, and the graph is a horizontal line, $Q=1500$. Integrating this, $M=1500x+C$; and as there can be no bending moment at the free end, $M=0$ when $x=6$. Hence $C=-9000$, and $M=1500x-9000$ (Fig. 190).

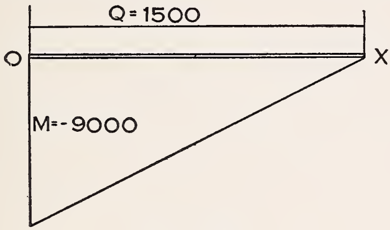


FIG. 190.

It is easy to see that the bending moment at the wall due to a load of 1500 pounds six feet out should be numerically 9000 pound-feet. Actually the sign agrees with the convention of Art. 97; but it is usually a little safer to determine the constant in the way we have done. The dangerous section is here obvi-

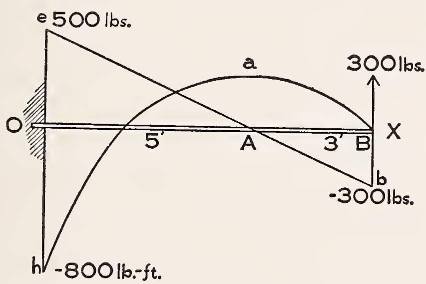


FIG. 191.

ously at the wall, and the beam must be strong enough to withstand a B. M. of 9000 pound-feet.

Example 2.—Consider a cantilever beam 8 feet long, carrying a uniformly distributed load of 100 pounds per foot run. A vertical cable, attached to the outer or free end, is tightened until it sustains a load of 300 pounds (see Fig. 191). To

find the dangerous section, note that since 300 pounds of the total load is carried at B, 500 pounds is carried at the wall, O. Hence $Q=500-100x$, giving eb for the S. F. curve. Then $M=500x-50x^2+C$; and since $M=0$ for $x=8$, we find $C=-800$. Thus $M=500x-50x^2-800$. Now the highest point on the bending-moment curve occurs at a, where $Q=0$; that is, where $x=5$ feet. This gives $Aa=+450$ pound-feet, which is the greatest *positive* value of M. But at the *wall* we have $M=-800$ pound-feet, which is *numerically* greater than the positive value Aa; hence the dangerous section is at the wall.

Problems

320. Find the maximum B. M. in Prob. 317.

321. Two locomotive wheels, $5\frac{1}{2}$ feet apart and weighing 25,000 pounds each, roll over a simply supported beam, span 30 feet. Find position of wheels for greatest B. M. and the position for the greatest reaction.

322. Find the maximum B. M. in Prob. 319.

102. Distribution of Stress.—We have hitherto considered only the total effect of the stress across any vertical section of a loaded beam without inquiring how the stress is distributed. In the case of shearing stress we shall not make that inquiry, for the reason that it is only in beams that are relatively very short that failure by shear is a possible cause of rupture. For beams of all ordinary shapes, failure will occur by bending long before any dangerous shearing stress is produced. Failure by bending is due to an excessive tensile or compressive stress in some part of the section. To determine the distribution of this stress we must take account of the nature of the strain produced by the loading; that is, the actual distortion of the beam from its original form. This distortion in a properly designed beam is relatively very small; but its nature and magnitude must be considered for the purpose, among others, of determining the distribution of the normal stress.

103. Bending.—When any part of a beam is acted on by a bending moment but no vertical shearing force, it is said to be subject to *simple* bending, *e. g.*, the middle section of the beam in Fig. 192.

In this case we assume (Bernoulli's assumption) that when the beam is bent, transverse sections originally plane remain plane, and also remain perpendicular to the longitudinal fibers. The word "fiber" here refers to a prism of infinitesimal cross section originally parallel to the axis of the beam. There is no implication that the material of the beam is of a fibrous nature.

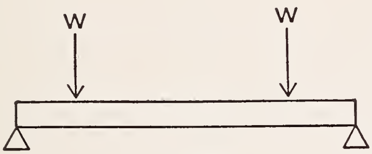


FIG. 192.

It follows from this assumption that the fibers on the concave side are shortened and so subjected to a compressive stress and those on the convex side lengthened and hence under tensile stress. Somewhere between these there must be a surface in which the fibers are neither elongated nor compressed. This is called the *neutral surface*. A vertical plane through the original position of the axis of the beam is called the *plane of bending*. The intersection of this with the neutral surface is called the *elastic curve*. Any transverse section meets the neutral surface in a line which is called the *neutral axis* of that section. If, as is commonly the case, the beam is of uniform cross section throughout its length, the neutral surface will be plane before bending, cylindrical after, and always perpendicular to the plane of bending.

Whatever the shape of the elastic curve as a whole, it will have at any point a center of a curvature, and any short arc of it will be very closely approximated by a circle with this center (Fig. 193). Any two transverse sections cut the plane of bending in radii of this circle. Let its radius be R. Then any arc of the circle which subtends an angle θ is of length $R\theta$. Any other longitudinal fiber between the same two transverse sections has the same original length. If it is at a distance y from the neutral surface its strained length will be $(R \pm y)\theta$ according as it is toward the convex or concave side of the neutral surface. In either case its change of length is $y\theta$ and this divided by $R\theta$ the original length gives y/R as the strain. Hence by Hooke's Law, the stress in the fiber is $p = Ey/R$. Thus the stress in any fiber is proportional to its distance from the neutral surface.

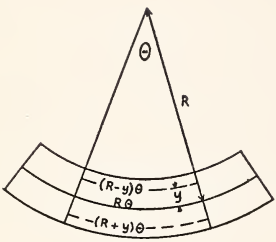


FIG. 193.

104. Location of the Neutral Axis.—We have already seen (Art. 96) that the total horizontal component of the normal forces across any section is zero. But since the differential element of this force is $p \cdot dA$, the total force is $\int p \cdot dA$ or $\int E/R \cdot y \cdot dA$ over the section. Note that y will be positive on one side of the neutral axis and negative on the other, and that corresponding to this change of sign the normal force acts in opposite directions on the two sides of the neutral axis (see Fig. 194).

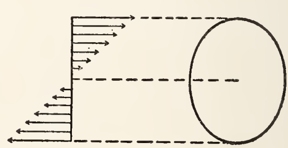


FIG. 194.

Since E and R are constant for the integration, we have

$$\frac{E}{R}\int y\cdot dA=0. \quad \text{But } \int y\cdot dA=\bar{y}\cdot A,$$

where $\bar{y}$ is the distance from the neutral surface to the center of area of the section. Hence $\bar{y}=0$, or *the neutral axis of any section passes through its center of area.*

105. Moment of the Internal Couple.—The total moment about the neutral axis of the normal force across the section will be

$$\int y\cdot p\cdot dA=\frac{E}{R}\int y^2 dA=\frac{E}{R}\cdot I,$$

where I is the so-called "moment of inertia" of the area of the section about its neutral axis. Since any portion of the beam is in equilibrium, the moment of the couple formed by the internal stress across any section must be balanced by the moment of the external forces on one side of the section. This last we have studied in the preceding articles and called M. Hence we have $M=EI/R$. Combining this with the result of Art. 103, we have

$$\frac{p}{y}=\frac{M}{I}=\frac{E}{R} \tag{5}$$

These are the most important and fundamental formulæ in the theory of bending. The student should make quite sure before proceeding that he has clearly in mind the exact meaning of each of the six letters involved. The value of the second fraction is known at any cross section if we know the loading of the beam and the dimensions of the cross section. Equating this to the first fraction enables us to determine the stress at any distance from the neutral axis, in particular at the extreme fiber, *i. e.*, the farthest from the neutral axis. The relation $M/I=E/R$ enables us when we know the material of the beam (which de-

termines E) to calculate the degree to which the beam is distorted, or its curvature.

106. Units of Measurement.—In using the equation of the preceding article care must be taken to be consistent in the use of units. We commonly measure p in pounds per square inch; therefore y should be in inches. The quantity I used here must be carefully distinguished from what is properly called the moment of inertia. The latter involves density or mass, and has the dimensions ML^2. The I used here is $\int y^2 dA$ and consequently has the dimensions L^4, being quite independent of density and depending only on the dimensions of the section. As the latter are regularly given in inches, I will be in inches. To be consistent with this, then, M must be in pound-inches.

Example 1.—What will be the greatest normal stress in a cantilever beam 8 feet long, 6 inches deep, and 2 inches wide produced by a weight of 500 pounds at its outer end?

Solution.—Here the maximum M is at the wall section and is

$$8 \cdot 500 = 4000 \text{ lb.-ft., or } 48000 \text{ lb.-in.}$$

The neutral axis of a section is its horizontal diameter. Hence for a rectangle about a diameter

$$I = \tfrac{1}{12} bh^3 = \tfrac{1}{12} \cdot 2 \cdot 216 = 36 \text{ in.}^4.$$

Hence

$$p/3 = 48000/36, \text{ or } p = 4000 \text{ lbs. per sq. in.}$$

Example 2.—A simple beam 8 feet long and 4 inches wide is to support a uniform load of 600 pounds per foot run. What must its depth if the stress is not to exceed 2000 pounds per square inch?

Solution.—The total load is 4800, and hence each supporting force is 2400.

$$Q = 2400 - 600x,$$
$$M = 2400x - 300x^2.$$

When $x=4$, $M=4800$ (the maximum value) in pound-feet or 57,600 in pound-inches,

$$I=\tfrac{1}{12}(4h^3)=\tfrac{1}{3}h^3,$$
$$y=h/2 \text{ and } p=2000.$$

Therefore,

$$2000/(h/2)=57600/(\tfrac{1}{3}h^3);$$
$$h=\sqrt{(3\cdot 57600)/(2\cdot 2000)}=\sqrt{43.2}=6.57 \text{ in.}$$

Problems

323. How much stronger is a $2'' \times 10''$ joist placed on edge than a square beam of the same material and equal cross section?

324. Find the dimensions of the strongest beam that can be sawed from a log of diameter d. Compare the strength of this beam with that of the original log.

325. A wooden cantilever beam is 2 inches wide, 5 inches deep and 8 feet long. How great a weight can it support at its free end, allowing a factor of a factor of safety of 8

326. A steel engine shaft of constant cross section rests on bearings 6 feet apart and supports a 10-ton flywheel midway between the bearings. Find the diameter of the shaft, allowing a factor of safety of 10.

327. The interior and exterior radii of a hollow cylindrical shaft are r_1 and r_2 respectively. Compare the strength of this shaft with that of a solid shaft of the same material and equal cross section. Compare the strength of it with that of a solid shaft of the same external diameter.

328. A simple wooden beam 3″ wide, 6″ deep and 15′ span has a load of 150 pounds at the middle. Compute the factor of safety.

329. A simple wooden beam 6″ wide, 8″ deep, span 10′, carries two equal concentrated loads at 2 feet and 8 feet from the left end. Find these loads for a factor of safety of 8.

330. A rectangular wooden beam 8 feet long, 10 inches deep, supported at the ends, carries loads of 3 tons and 2 tons at 2 feet and 5 feet respectively from the left end. If the maximum stress allowed is $\frac{1}{2}$ ton per square inch, find the width of the beam.

331. A steel beam 12 feet long, 6″ wide, and 1″ deep is fixed at the left end and unsupported at the right and weighs 20 pounds

per foot run. What weight at the free end will destroy the beam, if the ultimate stress is 100,000 pounds per square inch?

107. Section Modulus.—Equation (5) may be written $\frac{M}{p}=\frac{I}{y}$ If y has the value, usually called c, for the extreme fiber, we get the ratio I/c called the *section modulus.* This is a quantity depending entirely on the shape and size of the cross section, which gives at once the ratio of the bending moment at any section to the maximum stress in that section. It can be found worked out in handbooks for a number of standard sizes and shapes of beams. In a practical problem of design, M and p are known. The section modulus is obtained from the relation $I/c=M/p$. It is then merely a question of picking out of the handbook the proper size for the required section modulus.

Example.—A simply supported steel I-beam, span 20 feet, carries a uniform load of 500 pounds per foot run. What should be its depth?

Solution.—The greatest $M=500\cdot 20\cdot 20\cdot 12/8$ pound-inches. If a factor of safety 5 is used $p=13{,}000$ pounds per square inch. Then

$$\frac{I}{c}=\frac{500\cdot 20\cdot 20\cdot 12}{8\cdot 13{,}000}=23.08.$$

Reference to a steel handbook gives the proper depth as 9 inches with weight 35 pounds per foot.

The weight of the beam gives an additional uniform load. To check the result the load per foot run must be taken as 535 pounds. Then

$$\frac{I}{c}=\frac{535\cdot 20\cdot 20\cdot 12}{8\cdot 13{,}000}=24.7.$$

Since the section modulus of the 9-inch beam is 24.8 and thus larger than required, the design will not have to be changed.

Problems

332. What should be the section modulus of an I-beam, 20 feet long, which supports a uniform load of 450 pounds per foot run and two concentrated loads of 1500 and 2700 pounds at 6 and 14 feet respectively from the left end?

333. Wooden floor joists of 16-foot span and spaced 14 inches from center to center, are required to support a floor load of 100 pounds per square foot. What should be the size of the joists if the allowable stress is not to exceed 1100 pounds per square inch?

334. A balcony is to project 5 feet from a wall and be supported by wooden beams spaced 2 feet from center to center. If the floor load is to be 250 pounds per square foot and the fiber stress must not exceed 900 pounds per square inch, find the most suitable size for the beams.

335. An 18″ I-beam, weighing 90 pounds per foot, carries a uniform load of 50 tons. Compute its factor of safety if the span is 6 feet; also if the span is 9 feet.

336. A steel channel, span 20 feet, flanges horizontal, carries a uniform load of 13,500 pounds. What should be its size?

337. What size I-beam should be used to support the loads in Prob. 321?

108. Superposition.—If a beam is subjected to any particular loading, and the bending moment, or stress, etc., worked out at any point, and then the same thing done for another loading, it will be found that if both loads are applied at once we must at every stage of the work add together the two results previously found. This method is called *superposition*. It has its most important application in the use of a handbook. There a number of simple loadings are worked out, and one may solve a great many problems by treating them as combinations of these cases.

Problems

338. If the beam in Prob. 318 weighs 20 pounds per foot run and is loaded as stated, draw the B. M. and S. F. diagrams.

339. A cantilever beam, 8 feet long and weighing 5 pounds per foot run, carries a concentrated load of 300 pounds at the free end and another concentrated load of 200 pounds at its midpoint. Draw the B. M. and S. F. diagrams.

340. A steel girder, 18 feet long and weighing 40 pounds per foot run, is supported at its ends and carries concentrated loads

of 4000, 1000, and 600 pounds at 5, 9, and 13 feet respectively from the left end. Draw the B. M. and S. F. diagrams.

341. A beam 20 feet long, supported at its ends, carries a uniform load of 60 pounds per foot run, and also 800 pounds at 6 feet from left end and 500 pounds at 16 feet from left end. Draw the S. F. and B. M. curves and give the values of S. F. and B. M. at 14 feet from left end. Find the greatest B. M.

109. Torsion.—If a cylindrical bar is held fast at one end and a couple applied to it at the other, tending to twist the bar about its axis, the bar is said to be subject to torsion. The twisting moment tends to rotate adjacent sections of the bar in relatively opposite directions about the axis. Thus the stress acts tan-

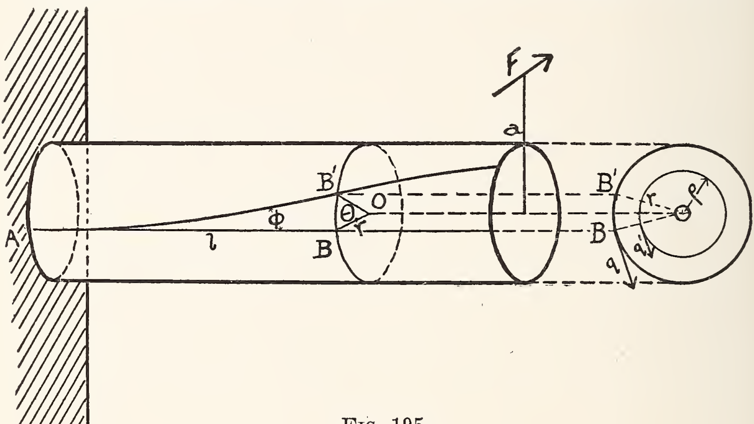


FIG. 195.

gentially on the section of the bar, and, therefore, torsion is a kind of shear. We shall confine our attention to bars with circular cross sections.

Suppose a force F with arm a tends to turn a shaft of radius r about its axis as shown in Fig. 195. Take a point B on the shaft and suppose a perpendicular section to be passed through it. To keep the part of the shaft to the right of the section in equilibrium, the action of the external forces must be resisted

by the internal stresses acting at the section. Hence, taking moments about the axis, the sum of the moments of the external forces acting on the shaft is equal to the sum of the moments of the internal forces. The former, in this case $F \cdot a$, is called the *torque* and denoted by T; the latter is called the resisting moment. Thus we say

Torque = Resisting moment.

If in Fig. 195 an element of the cylinder AB is twisted into the curve AB', we will denote the angle BAB' by ϕ and the angle BOB' by θ. The angle ϕ is called the shearing strain; θ is called the angle of twist. We find by experiment that $q = G\phi$, where G is a constant and is called the "shearing modulus of elasticity" or "modulus of torsion." From the figure we see that $l\phi = r\theta$. Thus $r = \frac{ql}{G\theta}$, and we see that the stress is proportional to the distance from the axis and reaches its maximum value at the surface. Thus if q' is the stress at a distance ρ from the center, $q/r = q'/\rho$ and $q' = \rho q/r$. On a circular ring of radius ρ and width $d\rho$ the total stress is $q' 2\pi\rho d\rho$. Taking moments about O and integrating over the whole area, we find the resisting moment. Thus

$$T = \int_0^r 2\pi q' \rho^2 d\rho = \frac{2\pi q}{r} \cdot \int_0^r \rho^3 d\rho = \frac{\pi r^4 q}{2r}.$$

But $\pi r^4/2$ is the "Polar Moment of Inertia" of the area of the section about O and we may write

$$T = q/r \cdot I_p,$$

which is analogous to the formula for Bending Moment $M = p/y \cdot I$.

Substituting the value of q in terms of θ and solving for θ we have

$$\theta = \frac{lT}{GI},$$

which shows that the angle of twist is proportional to the twisting moment.

Example.—What twisting moment in ton-inches can a steel shaft 3 inches in diameter transmit safely? Use a factor of safety 10.

Solution.—From the table

$$q = 50000 \text{ lbs. per sq. in.}$$
$$I_p = \pi\left(\tfrac{3}{2}\right)^4 \cdot \tfrac{1}{2} \text{ in inch units.}$$

Therefore

$$T = \frac{\frac{1}{10}(50{,}000)\frac{1}{2}\pi\left(\frac{3}{2}\right)^4}{\frac{3}{2}} = 26{,}500 \text{ lb.-in.}$$
$$= 11.8 \text{ ton-inches.}$$

Problems

342. Compare the torsional strength of a hollow shaft of radii r_1 and r_2 with that of a solid shaft of the same cross-sectional area.

343. If $r_1 = \frac{7}{8} \cdot r_2$ compare the strengths of the two shafts in the above problem.

344. What torque would be required to twist off a steel spindle 1 inch in diameter?

345. Compute the stress in a hollow shaft, outside diameter 18″, inside diameter 10″, subjected to a torque of 250 ton-feet.

346. A solid round shaft is subjected to a torque of 300 ton-feet. Find the diameter if the stress is limited to 6000 pounds per square inch.

347. A steel wire 0.18″ in diameter and 20″ long is twisted through an angle of 18°.5 by a moment of 20 pound-inches. Find the shearing modulus of elasticity.

Review Problems

348. A stone pillar has to carry a load of 8 tons and its cross section is to be a rectangle whose sides are in the ratio 3 : 4. If the stress allowed is 120 pounds per square inch, find the dimensions of the smallest section which will suffice.

349. A steel rod 2″ in diameter and 8 feet long stretches $\frac{1}{10}$″ when subjected to a pull of 44 tons. Find the modulus of elasticity.

350. A steel tube (inner diameter 8 inches, outer diameter 9 inches) is 5 feet long and is subjected to a pull which lengthens it $\frac{1}{10}$ inch. Find the pull in tons.

351. A hollow steel column is 8′ long, 10″ square outside, and square in section within. Under a tensile force of 700 tons it stretches $\frac{1}{10}$″. Find the thickness of the walls of the column.

352. What is the length of an iron rod (vertical) which will just carry its own weight? ($p = 7000$ pounds per square inch.)

353. The steel plates of a girder are 1″ thick and are double riveted with double butt straps. Rivets are $\frac{3}{4}$″ in diameter. If the shearing stress is $\frac{5}{6}$ the tensile stress, find the pitch for equal strength. Also find the efficiency of the joint.

354. A copper pipe, 2″ internal diameter, $2\frac{1}{2}$″ external diameter, is subjected to an internal pressure of 1200 pounds per square inch. Find the stress in tons per square inch.

355. A beam 4″ wide and 8″ deep is subjected to a bending moment of 160 ton-inches; what is the maximum fiber stress?

356. A steel rod, 10 feet long, 3 inches in diameter, is supported at its ends; what is the greatest central load it can carry, if stress is limited to 4 tons per square inch?

357. A T-beam, 12 feet long and with section shown in Fig. 196, carries a central load of 24 tons; neglecting the weight of the beam, find the maximum fiber stress.

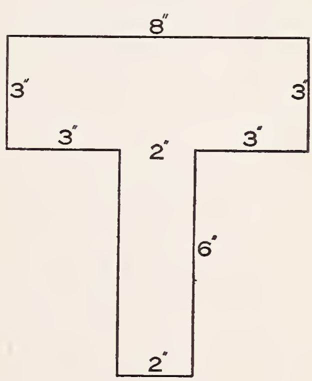


FIG. 196.

358. What is the radius of the smallest circle into which an iron rod $1\frac{1}{2}$″ in diameter can be bent without injury, the stress being limited to $3\frac{1}{2}$ tons per square inch?

359. A log is in the form of a triangular prism, its cross section being an equilateral triangle, 30″ on a side. Find the breadth and depth of the rectangular beam of greatest strength which can be sawed from this log.

360. A steel I-beam is 24 feet long, and carries 28000 pounds at its center. Its flanges are $7''$ by $\frac{4}{5}''$ and the web is $24''$ by $\frac{1}{2}''$. If the weight of the beam is neglected find the maximum fiber stress.

361. A plank 15 feet long, supported at its ends, carries 400 pounds 4 feet from the left end, 300 pounds 8 feet from the left end, and 250 pounds 10 feet from the left end. Draw the S. F. curve and find the S. F. 7 feet from the left end.

362. A beam 16 feet long, supported at its ends, carries a uniform load of 80 pounds per foot run. Draw the S. F. curve and give the value of the S. F. at 12 feet from the left end.

363. A beam 10 feet long has its left end fixed in a wall and supports a load of 1200 pounds at its free end. Draw the S. F. curve and find the value of the S. F. at 10 feet from the left end.

364. A beam 16 feet long, supported at its ends, carries a load of 60 pounds per foot run, and has concentrated loads of 1 ton and 3 tons at 4 feet and 10 feet, respectively, from the left end. Draw the S. F. curve, evaluate the S. F. at 6 feet from the left end and find where the S. F. $= -5000$ pounds.

365. An oak beam 12 feet long, 18 inches wide and 12 inches deep, floats in sea water. It is loaded at the center with a weight which just immerses it wholly. What is the greatest value of the shearing force and where does it occur?

366. A beam 12 feet long has its left end fixed in a wall, and it carries a uniform load of 80 pounds per foot run, together with a concentrated load of 800 pounds 7 feet from the wall. Draw the S. F. curve and evaluate the S. F. at 8.5 feet from the wall.

367. A beam 12 feet long, supported at its ends, carries a load of 1600 pounds 9 feet from the left end. Draw the S. F. and B. M. curves, give the value of B. M. at the center and the greatest value of B. M.

368. A beam 16 feet long, supported at its ends, carries a uniform load of 80 pounds per foot run. Draw the S. F. and B. M. curves, and give B. M. at the center and 14 feet from left end.

369. A beam 12 feet long, fixed at left end and unsupported at the right, carries a uniform load of 100 pounds per foot run. Draw the S. F. and B. M. curves and give the maximum value of B. M. and its value 8 feet from left end.

370. A steel bar, 4 feet long, 1 inch wide, is supported at its ends. It has a weight of 200 pounds at its center and one of 600 pounds at 3 feet from the left end. Neglecting the weight of the bar, find its depth if stress is limited to 12,000 pounds per square inch.

371. A steel beam is 12 feet long, 8 inches square, supported at its ends, and carries the following loads: 24 tons, 4 tons, and 20 tons at 5 feet, 7 feet, and 10 feet, respectively, from the left end. Neglecting the weight, find the maximum stress due to bending.

372. A beam, 13 feet long, supported at the ends, carries loads of 1000 pounds, 800 pounds, and 1140 pounds at 3 feet, 8 feet, and 10 feet, respectively, from the left end. Draw the S. F. and B. M. curves and find the B. M. 9 feet from the left end.

373. A beam 20 feet long, supported at the ends, carries a load which uniformly increases from 20 pounds per foot run at the left end to 60 pounds per foot run at the right end. Draw the S. F. curve, find the maximum value of the S. F. and the point where the S. F. $= 0$.

374. A wooden beam of rectangular section is 8 inches wide; find its depth, if the maximum B. M. is 18 ton-feet and the stress allowed is $\frac{3}{4}$ ton per square inch.

375. What is the diameter of the smallest circle with which $\frac{1}{4}$-inch steel wire may safely be coiled, keeping the stress within 5 tons per square inch?

CHAPTER VIII

RECTILINEAR MOTION

110. Velocity.—Previous chapters have dealt mainly with bodies at rest, any mention of motion being incidental. This and the succeeding chapters will deal with motion and moving bodies, beginning in this chapter with motion in a straight line. The *average velocity* of a body moving in a straight line during a given time interval is defined as the distance moved divided by the time elapsed. If this quotient is the same, regardless of the choice of the interval, the velocity is said to be constant. If this is not the case, we define the velocity at any instant to be the limiting value of the average velocity for an interval beginning with that instant, as the length of this interval decreases and approaches zero. If s is the whole distance traversed in the time t, then the average velocity is the increment of distance Δs divided by the increment of time Δt, and the instantaneous velocity is $\lim\limits_{\Delta t=0} \dfrac{\Delta s}{\Delta t}$ or $\dfrac{ds}{dt}$, the derivative of s with respect to t.

The numerical measure of the velocity depends, of course, upon the units of distance and time employed, and the unit of velocity must name both of these, *e. g.*, miles per hour, or feet per second. If the unit of length is changed, the numerical measure of the velocity changes in the same way as the measure of the length; that is, since the number of feet is multiplied by 12 to get the number of inches, the number representing the velocity in inches per minute is 12 times that in feet per minute. A change in the unit of time, however, produces the opposite effect. Thus in changing minutes to seconds we multiply by 60; but in changing feet per minute to feet per second, we must divide by 60. We say then that velocity is of dimensions 1

in length and -1 in time, L^1T^{-1}, and the units are commonly written in the corresponding form, mi/hr, ft/sec, etc.

Problems

376. How many feet per second is a velocity of 15 miles per hour?

377. A piece of wood floating on a river takes 5 seconds to pass under a bridge 14 yards wide. Find the speed of the stream in (a) feet per second, (b) miles per hour.

378. It takes a stone 3 seconds to fall 154 feet. What is its average velocity in miles per hour during that time?

379. Express a velocity of 1094 yards per hour in centimeters per minute.

380. Express a velocity of $7\frac{1}{3}$ feet per second in kilometers per hour, given 1 meter $=3.281$ feet.

381. If telegraph poles by the side of a railway are a yards apart and n poles are passed in t minutes, find the velocity of the train in miles per hour.

382. If the distance in feet passed over by a moving body in t seconds is given by the equation

$$s=3t^3+4t+2,$$

find the velocity at the end of 3 seconds, 2 seconds, $\frac{1}{3}$ second.

383. If the distance is given by the equation

$$s=10\cos\frac{\pi}{2}t,$$

find the velocity when $t=0, \frac{1}{3}, 1, 1\frac{2}{3}, 2, 2\frac{1}{3}, 3, 3\frac{2}{3}, 4$.

111. Acceleration.—If the velocity v is a variable, the change in velocity Δv in a time interval Δt, divided by this Δt, is the average acceleration. The limit of this as Δt approaches zero is *the acceleration* at the instant at which Δt begins, and is therefore equal to dv/dt or d^2s/dt^2. It is regularly denoted by the letter a. The unit of acceleration must be a unit of velocity (or change in velocity) per unit of time, and hence the unit of time occurs twice, both times in the denominator. The time units need not be the same in the two cases, but frequently are.

Thus if a train at one instant is moving with a speed of 30 mi/hr and 5 minutes later with a speed of 45 mi/hr, its change of velocity is 15 mi/hr, but, as this has occurred in 5 minutes, the average change per minute is 3 miles per hour; that is, the average acceleration is 3 miles per hour per minute. If the length unit or either of the two time units is changed the measure of the acceleration is changed accordingly. Thus 3 miles per hour per minute is 15,840 feet per hour per minute, or 264 feet per minute per minute, or 4.4 feet per minute per second, or .073 feet per second per second. Note that the measure of the acceleration in feet per minute per minute must be divided by the *square* of 60 to obtain the acceleration in feet per second per second. The dimensions of acceleration, therefore, are L^1T^{-2} and the units may be written ft/sec^2, cm/sec^2, etc.

Problems

384. Express an acceleration of 800,000 feet per hour per hour in miles per minute per minute.

385. An acceleration of 27,000 miles per hour per hour is how many feet per second per second?

386. If a point acquires in 15 seconds a velocity of 30 miles per hour, what is the acceleration in feet and seconds?

387. A body has a velocity of 33 miles per hour; 3 minutes later its velocity is 60 miles per hour. Express the average acceleration in feet and seconds.

388. If s, the distance traversed by a moving body in time t, is given by the equation

$$s=at+bt^2+ct^3,$$

find velocity and acceleration at time t.

389. When s and t are connected by the equation

$$s=a\cos(b+kt),$$

find the velocity and acceleration at time t.

390. If a point moves so that $s=\sqrt{t}$, show that the acceleration is negative and proportional to the cube of the velocity.

391. If a point moves so that in t seconds $s=10 \log \frac{4}{t+4}$ feet, find the velocity and acceleration at the end of 1 second. At the end of 16 seconds.

392. The motion of a particle in a straight line is expressed by the equation $s=5-2\cos^2 t$. Express the velocity and acceleration at any point in terms of s.

112. A very important form for the acceleration in terms of derivatives is obtained by the elimination of t, thus

$$a=\frac{dv}{dt}=\frac{dv}{ds}\cdot\frac{ds}{dt}=\frac{dv}{ds}\cdot v.$$

Hence

$$\int a\cdot ds=\tfrac{1}{2}v^2+C=\tfrac{1}{2}(v^2-v_0^2),$$

where v_0 is the velocity when $s=0$, or

$$v^2=v_0^2+2\int_0^s a\cdot ds.$$

If a is a constant, this becomes

$$v^2=v_0^2+2as.$$

It must be borne in mind, however, that this last expression applies only when the acceleration is constant.

Problems

393. A body starts with a velocity of 4 feet per second and moves with an acceleration of 1 foot per second per second. In what time will it acquire a velocity of 30 miles per hour?

394. A body starting from rest, and moving with uniform acceleration, describes 171 feet in the tenth second. Find its acceleration.

395. A point starts with a velocity of 100 centimeters per second and moves under a retardation of 2 centimeters per second per second. When will its velocity by zero, and how far will it have gone?

396. A point moving with uniform acceleration describes in the last second of its motion 9/25 of the whole distance. If it started from rest, how long was it in motion and through what distance did it move if it described 6 inches in the first second?

397. A point moving with uniform acceleration describes 25 feet in the half-second which elapses after the first second of its motion, and 198 feet in the eleventh second of its motion. Find the acceleration and initial velocity.

398. A point starts from rest and moves with a uniform acceleration of 18 feet per second per second. Find the time taken by it to traverse the first, second, and third foot respectively.

113. Relative Velocity.—In determining the position, velocity, or acceleration of a body, we must always measure these with respect to some body, some system of coordinates or the like, which for the time being we regard as fixed. Thus if a man walks from one end to the other of a car 40 feet long in 10 seconds, we should say that he walks at the rate of 4 feet per second. This is his speed with reference to the car. If, however, the car itself is moving, his speed with reference to the track is quite different. It is easy to see that the velocity of the man with respect to the track is the algebraic sum of the velocity of the car with respect to the track and that of the man with respect to the car. On the other hand, if the velocities of both are given with respect to the track, the velocity of the man relative to the car is the algebraic *difference* of their velocities, that of the car (*i. e.,* the new reference body) being subtracted. We can formulate the principle illustrated by the first case in the equation, velocity of A relative to $C=$velocity of A relative to $B+$velocity of B relative to C, proper attention being given to the signs of the velocities. If we solve for the first term of the second member, we get the theorem illustrated by the second case, velocity of A relative to $B=$velocity of A relative to $C-$velocity of B relative to C. The same principle applies to relative accelerations.

Problems

399. A train is moving with a speed of 30 miles per hour, and another train on a parallel track is going in the opposite direction with a speed of 20 miles per hour. What is the velocity of the second train as observed by a passenger in the first?

400. A man can throw a certain body with a velocity of 20 feet per second. If he stands on the rear platform of a train moving 30 miles per hour and throws the body straight backward, what velocity will it have with respect to the track when it leaves his hand?

114. Mass and Momentum.—So far in this chapter we have spoken only of motion without reference to the cause of the motion or the character of the moving body. We shall now introduce the idea of *mass.* Like many other elementary concepts, it is not easy to define satisfactorily in a few words. A very rough notion of it is obtained by describing it as quantity of matter; but this suggests no definite method for its measurement. The most fundamental fact about it is this: If a *body* is defined as a definite portion of matter, then the mass of a body remains unchanged if no material is added to the body or taken from it. This at once distinguishes mass from volume, which changes with alterations of pressure, temperature, or chemical state. It also distinguishes it, but less obviously, from weight, which is the measure of the earth's apparent attraction. For it is found that the weight of a given body as measured by a spring balance varies quite appreciably with changes of latitude and elevation above sea-level. This distinction must be made even if we confine our attention to bodies in the neighborhood of the earth's surface; and in astronomy the notions of mass and weight become entirely distinct, mass remaining fundamental, a "body-constant," and weight becoming quite accidental and variable, depending on the body's location. A second property of mass is that the mass of a body is equal to the sum of the masses of its component parts. For practical measurements of mass, we make use of a third assumption: that in neighboring localities near the surface of the earth, the masses of two bodies are equal if their weights are equal; that is, if they balance each other on the scale pans of an ordinary beam balance, it being a matter of experi-

ment and observation that two bodies which thus balance each other when weighed in one locality will do the same when weighed in any other locality. These three assumptions are sufficient for us to measure mass, *i. e.,* to express the mass of any body as a multiple of some standard mass taken as a unit. There are two principal standards of mass: the pound, preserved by the British government in London, and the kilogram, preserved by the French government, official copies being kept by the Bureau of Standards in Washington. It is to be noted that these are standards of mass and not of weight. For, although they are copied by being weighed, the copy when transported to a different region of the earth's surface may have an appreciably different weight; but its mass is assumed to be still equal to the mass of the prototype. The kilogram is 2.2046 pounds. Scientists commonly use as a standard mass the gram, or one-thousandth part of a kilogram.

It is possible to express all physical units in terms of three fundamental ones. There is some difference in practice as to which three shall be used; but the most usual selection is that of length, time, and mass. Every other unit then has "dimensions" in the three given units.

The *momentum* of a moving body is defined as the product of its mass and its velocity. Its dimensions, therefore, are $M^1L^1T^{-1}$.

115. Newton's Laws of Motion.—The most fundamental relations connecting space, time, mass, and force are expressed in the traditional form of three laws formulated by Sir Isaac Newton as follows:

I. Every body continues in its state of rest or uniform motion in a straight line, except in so far as it may be compelled by impressed forces to change that state.

II. Change of motion is proportional to the impressed force, and takes place in the direction of the straight line in which the force acts.

III. To every action there is always an equal and contrary reaction, or, the mutual actions of any two bodies are always equal and oppositely directed.

These laws are generalizations from experience and differ from any other physical laws only in being of the widest possible application. They are not altogether original with Newton, much of their content having been previously known to Galileo and Huyghens. They were originally stated in Latin; but the English translation given above has become more or less traditional. As a consequence, the laws require considerable elucidation for the modern student. They are included here partly as a matter of historic interest and general information. It is important, however, that the student should obtain a thorough understanding of their meaning as stated.

The First Law is sufficiently clear in its language; but at first sight it appears to contradict experience. We commonly find that it is necessary to apply a force if we are to keep a body moving uniformly in a straight line. Actually, however, this force is required only to overcome frictional or other resistances, as we can readily see if we observe that the force required becomes less and less as these resistances are diminished. Thus a heavy body sliding on smooth ice, or a large ship moving slowly in still water, will continue for some time with only a small reduction in speed, and even in these cases there is some resistance. All bodies in nature are subject to the action of *some* force; but if a body is acted on by a combination of forces whose resultant is zero, the First Law is applicable. This law has been paraphrased by Kelvin and Tait in the words, "It is force alone which can produce a change of motion." It is thus mainly preparatory to the Second Law, which tells us *how* force changes motion.

Regarded quantitatively, "change of motion" in the Second Law must be understood to mean change of *momentum,* not

merely change of velocity. Thus equal forces acting on unequal masses for a given time will produce unequal changes in their velocity, but equal changes in their momentum. It must be observed also that the forces must be assumed to act for the same length of time in order that the resulting changes in momentum shall be proportional to the forces.

The Third Law has to do with the nature of a force and states that it is something with two ends. That is, if one body is pushed upward with a certain force, there is somewhere another body which is being pushed downward with an equal force. Or, recalling the definition of a force as a push or a pull, we may say that a force is that which pushes two bodies apart or pulls them together. Thus, if a man's hand pushes against a table with a certain force, the table pushes against his hand with the same force and in the opposite direction, and this is true regardless of whether this force is sufficient to move the table or not. The equality of action and reaction must not be confused with the condition for equilibrium. A body is in equilibrium when it is acted on by two equal and opposite forces; but these two forces are *never* action and reaction. For they act on the *same* body, whereas action and reaction always act on *different* bodies. Action and reaction are sometimes spoken of as the two ends of a single force, but more commonly as two forces. This is merely a question of words, and is unimportant. It is very necessary, however, to remember the existence of the reaction, and, if we are considering the effect of a force on a system of bodies, to note whether the reaction acts on the system or on an outside body.

116. Conservation of Momentum.—If two bodies act on each other for a certain interval of time, we learn from the Third Law that the force acting on one is always equal and opposite to the force acting on the other, and hence by the Second Law the change in the momentum of one is equal and opposite to the

change in the momentum of the other. If we confine our attention to motion in one straight line, we may state this in the form: whatever momentum the one body gains the other loses, due regard being paid to algebraic sign, and the sign of momentum being the same as that of velocity. The combined momentum of the two bodies is then the same at the end of the interval as at the beginning. If we apply this principle to several bodies exerting mutual actions and reactions upon one another, but affected by no external force, we see that the action and reaction between any two bodies produce equal and opposite changes in momentum, and hence all the actions and reactions together leave the total momentum unchanged. We may then state the principle that *if a system of bodies is not acted upon by any external force, its total momentum is a constant.* This is known as the principle of conservation of momentum. Thus if we know that at one instant a mass of 20 pounds is moving east at a speed of 5 ft/sec, a mass of 10 pounds is moving east at 3 ft/sec and a mass of 2 pounds is moving west at 10 ft/sec, and if later, after certain collisions or other mutual actions and reactions, we know that the first body is moving east at 4 ft/sec and the second east at 2 ft/sec we may find the speed v of the third body by writing the equation: total original momentum equals total final momentum. Thus, taking east as the positive direction,

$$20 \times 5 + 10 \times 3 + 2(-10) = 20 \times 4 + 10 \times 2 + 2v;$$

whence $v = 5$, that is, the third body has a final velocity of 5 ft/sec east.

Problems

401. A body of mass 3 pounds moving 13 ft/sec overtakes a body of mass 2 pounds moving 3 ft/sec in the same straight line. If they coalesce and form one body, find its velocity.

402. A shot of mass 1 ounce is projected with velocity 1000 ft/sec from a gun of mass 10 pounds. Find the velocity with which the latter begins to recoil.

117. Elastic Impact.—If two more or less elastic bodies collide, it is found as a matter of experiment that, if the collision is not too violent, the velocity with which they recede from each other bears a ratio to the velocity of approach which is practically independent of the size and shape of the bodies and the velocities themselves, and depends only on the materials of which the bodies are made. This ratio is called the *coefficient of elasticity,* or, better, *the coefficient of restitution,* for the two substances and is commonly denoted by e. Its value varies between the limits zero and unity, being zero for perfectly inelastic substances such as putty, and nearly unity for two bodies both of glass. It will also be zero in the case of one body penetrating the substance of the other and remaining imbedded in it, as a bullet remaining imbedded in a block of wood. It is to be noted that there is no such thing as a coefficient of restitution for a single substance unless we understand that to mean the coefficient for the impact of two bodies both of that substance. Moreover, if we know the coefficient for two bodies of one substance and also for two bodies of some other substance, we cannot predict the coefficient for two bodies, one being of one of these substances and the other of the other. Thus (according to Jeans) the coefficient of restitution for iron on iron is about .66, for lead on lead, about .20; but for iron on lead the coefficient is not even intermediate between these values, but is about .14. There is here a decided analogy between the coefficient of restitution and the coefficient of friction, as the latter is also apt to be smaller for unlike substances than for like.

Since the velocity of approach is the algebraic difference in the velocities of the bodies before impact, and the velocity of separation the same difference after impact except that the subtraction is performed the opposite way, we may say that the relative velocity before impact must be multiplied by $-e$, to

obtain the relative velocity after impact. This fact, together with the principle of conservation of momentum, enables us to predict completely the motion of two bodies after impact if we know their velocities before and their coefficient of restitution.

Thus if m and m' are the masses of two bodies, u and u' their respective velocities before impact, and v and v' their velocities after impact, we have from the conservation of momentum

$$mu + m'u' = mv + m'v',$$

and from the definition of e

$$v - v' = -e(u - u').$$

Example 1.—A mass of 20 pounds moving north with a speed of 8 ft/sec meets a mass of 10 pounds moving south with a speed of 5 ft/sec and their coefficient of restitution is $\frac{1}{2}$. What are their velocities after impact?

Solution.—From the conservation of momentum

$$20v + 10v' = 20 \times 8 - 10 \times 5.$$

From the coefficient of restitution

$$v - v' = -\tfrac{1}{2}[8 - (-5)].$$

Hence,

$$v = \tfrac{3}{2} \text{ f/s and } v' = 8 \text{ f/s}.$$

This shows that the first body has its speed much reduced but continues in the same direction. The second body, however, has its motion reversed, as its original velocity was negative.

Example 2.—Find the velocities if the two bodies are perfectly elastic.

Solution.—The momentum equation is the same as before, but now $e = 1$, and hence

$$v - v' = -[8 - (-5)].$$

This gives

$$v = -\tfrac{2}{3} \text{ f/s}, \; v' = 12\tfrac{1}{3} \text{ f/s}.$$

In this case both bodies have their direction of motion reversed.

Problems

403. A ball A overtakes a ball B moving in the same straight line. A has half the mass and seven times the velocity of B. The coefficient of restitution between them is $\frac{3}{4}$. Find their velocities after impact.

404. A ball of mass 2 pounds impinges directly on a ball of mass 1 pound which is at rest. If the velocity of the former before impact is equal to the velocity of the latter after impact, find the coefficient of restitution.

405. Two balls have masses 14 and 7 pounds respectively, and velocities in opposite directions of 11 and 22 ft/sec respectively. The coefficient of restitution is $\frac{5}{6}$. Find their velocities after impact.

406. A ball drops from the ceiling of a room and after twice rebounding from the floor reaches a height equal to one-half that of the room. Find the coefficient of restitution.

118. The Equation of Motion.—In discussing Newton's Second Law it was tacitly assumed that the forces acting remained constant during the experiment. We shall generalize the law so as to remove this restriction, still confining our attention as yet to motion in a straight line. If we are to take account more explicitly of the time element, it is necessary to say that the change in momentum per unit of time is proportional to the force applied, the force remaining constant. Thus if t, v, m, and F denote time, velocity, mass, and force, respectively, we shall have from the Second Law

$$\Delta(mv)/\Delta t = \lambda F,$$

where λ is a numerical constant depending only on the units in which t, v, m, and F are measured. If F is variable this equation must become

$$\lambda F = \lim_{\Delta t = 0} \frac{\Delta(mv)}{\Delta t} = \frac{d(mv)}{dt} = m\frac{dv}{dt} = ma,$$

where m passes over the sign of differentiation because it is a constant. This result, that *mass times acceleration is a constant times the force applied,* is by far the most important and funda-

mental fact in dynamics. The rest of this chapter and much of what follows will consist of nothing but the development and illustration of this equation. The equation in this form, $\lambda F = ma$, is sometimes called the equation of motion, but more correctly the differential equation of motion, since a is a second derivative, d^2s/dt^2. The shorter phrase, equation of motion, is more properly restricted to the result of integrating this equation, s being obtained explicitly in terms of t.

119. Units of Mass and Force.—We must first discuss the determination of the constant λ for certain sets of units. We have already spoken of the units of mass and acceleration, but since the introduction of motion have said nothing about a unit of force. In statics, when we spoke of a force of one pound, we meant a force equal to the weight of one pound; that is, a force equal to that necessary to support a mass of one pound and keep it from falling. This must then be equal, if we take the surface of the earth as fixed, to the force which the earth's attraction exerts upon a mass of one pound. If this force is allowed to act without any other force being applied, we have the case of a freely falling body. It is a matter of experiment that for a given locality the acceleration in this case is a constant (for velocities small enough that we may disregard air resistance) and is denoted by g. It is equal to about 32 ft/sec² or 980 cm/sec². In the equation $\lambda F = ma$, we then have $a = g$. Also $F = m$, for the force exerted by gravity on a mass of 1 pound is a force of 1 pound. Accordingly, if the units are feet, seconds, and pounds of mass on one side of the equation and pounds of force on the other, λ will be equal to g, and the equation becomes $gF = ma$.

Problems

407. If $s = ae^{kt} + be^{-kt}$, show that the particle is acted on by a repulsive force which is proportional to the distance from the point from which s is measured.

408. If a particle moves so that

$$s = e^{-\frac{1}{2}ct} \cdot (a \sin ht + b \cos ht),$$

find expressions for the velocity and the acceleration. Hence show that the particle is acted on by two forces, one proportional to the distance from the origin and the other proportional to the velocity. Describe the motion of the particle.

120. Falling Bodies.—Within the limits of ordinary observation we find that a body allowed to fall freely in a vacuum has a constant acceleration g. The value of g varies appreciably with changes in latitude and elevation above sea-level. At sea-level the value in feet and seconds varies from about 32.09 at the equator to about 32.26 at the poles. We may regard 32.2 as correct to 3 significant figures in the temperate zone. The change with variation in altitude is smaller than this and not altogether regular. For certain purposes the value at latitude 45° and sea-level is regarded as standard. This is about 32.17.

Problems

409. A body is projected from the earth vertically with a velocity of 40 ft/sec. Find (1) how high it will go before coming to rest; (2) what time has elapsed when it is at a height of 9 feet.

410. A body moving vertically passes a point at a height of 54.5 cm. with a velocity of 436 cm/sec. With what initial velocity was it thrown, and how much longer will it rise?

411. A body moving downward passes a given point with a velocity of 50 meter/sec. How long before this was it moving upward at the same rate?

412. A tower is 288 feet high. One body is dropped from the top of the tower and at the same instant another is projected vertically upward from its base, and they meet half-way up. Find the initial velocity of the projected body and its velocity when it meets the descending body.

121. Absolute Units.—The equation $\lambda F = ma$ is of such fundamental importance that it is often desired to simplify it still further by a choice of units such that λ shall be equal to unity and the equation may read $F = ma$. Of the four units of length, time, mass, and force, we might choose any three arbitrarily, and then determine the fourth so as to make $\lambda = 1$. This is actually done in two principal ways. In both, the units of length and time are chosen without reference to this question. Then the unit of force is adjusted to that of mass or *vice versa.* In pure science the units of length, mass, and time are usually the centimeter, gram, and second (the C. G. S. system). We shall then have $\lambda = 1$ if we take as a unit of force that force which acting on a mass of 1 gram produces an acceleration of 1 cm/sec^2. This unit of force is called the *dyne.* It is the principal unit of force in all parts of theoretical physics, and is the basis for all the electric and magnetic units. The corresponding unit of force if we use feet, pounds, and seconds (the f. p. s. system) is the *poundal,* and is that force which acting on a mass of one pound produces an acceleration of 1 ft/sec^2. These units, the dyne and the poundal, are called *absolute* units of force, as distinguished from the pound force and the gram or kilogram force, which are called *gravitational* units because they are equal to the weight of the masses having the same names. The term absolute is used because the absolute units are the same in all localities, whereas the pound force and the gram force vary from place to place with the change in the earth's apparent attraction.

122. Engineer's Units.—Another method of making $\lambda = 1$ is that sometimes used by engineers. This is to use the gravitational unit of force and to change the unit of mass. The *engineer's unit of mass* is that mass which acted upon by a pound force has an acceleration of 1 ft/sec^2. The magnitude of this mass may be obtained by setting F and a equal to unity in

the equation $gF = ma$. This gives $m = g$; that is, the engineer's unit of mass is g times the ordinary unit or pound mass, or about 32 pounds. This unit of mass has no well-accepted name, being sometimes called merely the "engineer's unit," sometimes a "gee-pound," sometimes a "slug." Most people who use it rarely mention its name.

For many purposes in the use of the engineer's units it is not necessary to say whether the value of the pound force is the local value or some standard value. For accurate work, however, *e. g.*, in the computation of tables of steam pressure, it is necessary to define the units more exactly. This may be done by taking as a standard pound force the weight of a pound mass at sea-level and latitude 45°. The engineer's unit of mass will then be the pound mass multiplied by the standard value of g. In this case if the actual effect of gravity enters the problem it must be remembered that it is determined by the *local* value of g, and that in general the weight of a pound mass is in this system not exactly equal to a pound force. There seems to be no agreement among engineers on the exact definition of the pound force. Some use the local value, others a standard value. There is also some disagreement on the definition of this standard. Fortunately these discrepancies are in the third or fourth significant figure, and hence in many (though not all) practical problems they are negligible.

123. To sum up, we have three principal systems of units of force and mass:

1. The ordinary units, pounds mass and pounds force, or grams mass and grams force.

2. The absolute units, pounds mass and poundals, or grams mass and dynes.

3. The engineer's units, pounds force and slugs.

In (1) the equation of motion is $gF = ma$, in (2) and (3) it is $F = ma$.

In (2) both units are constant. In (1) the unit of mass is constant; the unit of force is the local pound and depends on the local value of g.

In (3) the practice of engineers varies, some using a local pound force and a mass unit to agree with it, others a standard pound force and a corresponding mass unit.

The units in (1) are those regularly used in every-day speech, those in (2) are regularly used by physicists. It must be understood that by no means all engineers use (3). Some use (1) and electrical engineers almost universally use (2).

The student should be prepared to understand statements and problems in any one of these systems of units, as he is likely to meet all of them in practice. He should also be able to convert any one into the other.

Example 1.—Show that a poundal is about equal to the weight of half an ounce.

Solution.—Since the result is wanted in ordinary units, we use the equation $gF = ma$. From the definition of a poundal $a = 1$ when $m = 1$. Hence $gF = 1$, and $F = 1/g$ in ordinary units, *i. e.*, $\frac{1}{32}$ of a pound or $\frac{1}{2}$ ounce.

Example 2.—How many dynes are in a force equal to the weight of 1 gram?

Solution.—As in the preceding, a force of 1 dyne is $1/g$ in ordinary units, *i e.*, 1/980 of a gram; or a gram weight equals 980 dynes.

Problems

413. Find the magnitude in dynes of the force which, acting on a kilogram for 5 seconds, produces in it a velocity of 1 meter/sec.

414. A force equal to the weight of a kilogram acts on a body continuously for 10 seconds and causes it to describe 10 meters in that time. Find the mass of the body.

415. A body of mass 200 tons is acted upon by a force of 112,000 poundals. How long will it take to acquire a velocity of 30 mi/hr?

124. Constant Force.—*Example 1.*—A mass of 10 pounds is acted on by a constant force of 25 pounds. If it starts with a velocity of 20 ft/sec in the same direction as the force acts, how far will it travel in the first three seconds? When will it have a velocity of 400 ft/sec?

Solution.—As the problem is stated in ordinary units, we have $gF=m\alpha$ where $g=32$, $F=25$, $m=10$. Hence $\alpha=80=dv/dt$. This is all there is of dynamics in the problem; the rest consists in integrating and determining constants. Thus $v=80t+C$. Since $v=20$ when $t=0$, we have $C=20$, and $v=80t+20=ds/dt$. Hence $s=40t^2+20t+C_1$; but $C_1=0$ because we measure the distance naturally from the starting point, *i. e.*, take $s=0$ when $t=0$. The equation of motion of this body is $s=40t^2+20t$. When $t=3$, $s=420$, *i. e.*, the body travels 420 feet in the first three seconds. To find when it has a velocity of 400 ft/sec put $v=400$ in the equation connecting v and t; then $400=80t+20$, $t=4.75$. Hence the body will take 4.75 seconds to acquire a velocity of 400 ft/sec.

Example 2.—Show that the distance traveled by a body starting with a velocity v_o and having a constant acceleration α, is equal to the algebraic sum of two distances, one the distance it would travel with a constant velocity v_o, and the other the distance it would cover if it started from rest with the acceleration α. Determine, by integration, the constants for each of the following three cases:

(1) Motion with constant velocity v_o. (2) Starting from rest with acceleration α. (3) Starting with velocity v_o and having acceleration α.

Example 3.—A mass of 40 grams is acted on by a constant force of 100 dynes. If it starts with a velocity of 250 cm/sec in the direction opposite to that in which the force acts, how far will it have gone when its velocity is reduced to 150 cm/sec? How far when it comes to rest?

Solution.—Using absolute units, the equation $F=m\alpha$ becomes $-100=40\alpha$, $\alpha=-2\frac{1}{2}$, $v^2=v_o{}^2+2\alpha s$, or $(150)^2=(250)^2-2(2\frac{1}{2})s$, or $s=8000$ cm. In the second case $v=0$ and $s=12500$ cm. Note that the equation $v^2=v_o{}^2+2\alpha s$ does not contain t, and is therefore particularly useful if t is neither given nor required.

Problems

416. A mass of 10 pounds falls 10 feet from rest, and is then brought to rest by penetrating 1 foot into some sand. Find the average force of the sand on it.

417. A bullet with initial velocity 1500 ft/sec strikes a target 1200 yards away with velocity 900 ft/sec. The path being supposed horizontal, find the ratio of the mean resistance of the air to the weight of the bullet.

125. In many problems it is possible to avoid altogether the question of units of mass. If in any system a force F acting on a body of mass m produces an acceleration a we have

$$\lambda F = ma.$$

If a new force F' acting on the *same* body produces an acceleration a', we have in the *same* system of units

$$\lambda F' = ma'.$$

Dividing one of these by the other

$$\frac{F}{F'} = \frac{a}{a'}.$$

If now we take the special case where the force F' is the weight of the body W, a' will be g, the acceleration of gravity, and we have

$$\frac{F}{W} = \frac{a}{g}.$$

Here it is only necessary that the weight W shall agree with F as regards the units in which it is measured and agree with g as regards the locality in which it is taken. This equation, written in the form

$$gF = Wa,$$

is substantially the same as that $gF = ma$ used in the system (1), if we remember that in that system mass and weight are numerically equal.

Problems

418. A force, acting on a mass of 10 pounds for 5 seconds, produces in it a velocity of 100 ft/sec. Compare the force with the weight of a pound, and find the acceleration it would produce in a ton.

419. A bullet moving 200 ft/sec is fired into a block of wood into which it penetrates 9 inches. If a bullet moving with the same velocity were fired into a similar piece of wood 5 inches thick, with what velocity would it emerge, supposing the resistance to be uniform?

420. A ball of mass 1000 grams is discharged with a velocity of 45000 cm/sec from a gun the length of whose barrel is 200 cm. Find the mean force exerted on the ball.

126. Relative Acceleration.—It should be observed that the value of g that we have spoken of so far is an acceleration measured relative to the surface of the earth taken as fixed. The acceleration of a falling body if referred to some other body regarded as fixed will be different, if the latter has an acceleration with respect to the surface of the earth. Thus if the supporting cable of an elevator should break, allowing the car to fall freely, the car and everything in it would have an acceleration g with respect to the earth. The acceleration of an unsupported body within the car would be g if referred to the earth, but zero if referred to the car. Thus an unsupported body w would remain at the same distance from the floor of the car as long as the car had the downward acceleration g, unless it were given an initial velocity with respect to the car. It is clear then that a body within the car would entirely lose its weight if suspended and weighed by an apparatus attached to the car.

Example 1.—A man weighing 160 pounds stands on the platform of a spring scale in an elevator. If the elevator starts up with an acceleration of 4 ft/sec^2, what will the scale register?

Solution.—The man's body is acted on by two forces, his weight of 160 pounds, and the upward pressure P exerted by the scale platform. The equation of motion then becomes

$$g(P-160)=160\times 4, \text{ or if } g=32,\ P=180.$$

Example 2.—Show that if the weighing apparatus is subject to any upward acceleration, the apparent weight of a body is increased by an amount which bears the same ratio to its weight at rest as the upward acceleration bears to the acceleration of gravity. Does this statement apply to a beam balance?

Solution.—Let the student proceed as in the foregoing example, using general values in terms of letters for each of the quantities involved. The statement does not apply to a beam balance, because the weights in both scale pans would be increased equally by the acceleration.

Problems

421. A man whose weight is 112 pounds stands in an elevator which moves with a uniform acceleration of 12 ft/sec^2. Find the force he exerts on the floor of the elevator when it is (1) ascending, (2) descending.

422. A balloon ascends with uniform acceleration so that a mass of 100 pounds exerts a force of 116 pounds on the floor of the balloon. Find the height of the balloon one minute after starting.

127. Motion on an Inclined Plane.—If a body rests on an inclined plane, and is not acted on by any force having an upward component sufficient to lift it off the plane, any motion that it may have will keep it in contact with the plane. If the forces and the initial velocity are all in a vertical plane perpendicular to this one, the motion will be rectilinear. In all cases of this sort the body is acted on by the force of gravity and also by the reaction or upward pressure of the plane, together with any other forces that may be present. If the plane is perfectly smooth, the reaction of the plane is normal to its surface. If it is not smooth the reaction may be resolved into two components, one normal and the other along the surface. Since the body remains in contact with the plane, no motion takes place in the direction normal to the surface, and hence the normal components of all the forces acting must be in equilibrium. If the other forces are known this determines the normal reaction of the plane, and from this we can determine the force of friction, if any.

Example 1.—A body is projected with an initial velocity of 80 feet per second up a perfectly smooth plane inclined at an angle of 30° to the horizontal. (1) How far from the starting point will it be at the end of 4 seconds? (2) How far up the plane will it go? (3) When will it return to its starting point? (4) What velocity will it then have? (Take $g=32$.)

Solution.—If the force of gravity W is resolved into two components normal to the plane and along it (Fig. 197) the former is balanced by the normal reaction of the plane, N; the latter, $W \sin 30°$, or $\frac{1}{2}W$, is the resultant force acting *down* the plane. The equation $gF=Wa$ then becomes $-\frac{1}{2}Wg=Wa$ or $a=-\frac{1}{2}g=-16$. Hence $v=80-16t$, and $s=80t-8t^2$. At the end of 4 seconds $s=80\times 4-8\times 16=192$ feet. The second question is to determine the maximum value of s. This of course occurs when its derivative v is zero. Then $0=v_0^2+2as$, $s=-v_0^2/2a=6400/32=200$ feet. The third question is to determine t when $s=0$ for the second time. This gives $80t-8t^2=0$, $t=0$, and $t=10$, the second root being the one required. The fourth question is answered by putting this value of t in the expression for v. This gives $v=80-160=-80$. The velocity is thus equal to the initial velocity, but in the opposite direction.

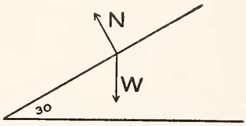


FIG. 197.

Example 2.—Show that if a body is projected with any initial velocity up a perfectly smooth plane, it will pass any point on the downward journey with the same speed that it passes the same point going up.

Solution.—Form the equation of motion with the general initial velocity v_0 and the inclination of the plane θ and then use the equation connecting v, a, and s in uniform acceleration.

Example 3.—Solve Example 1 with the plane having a coefficient of friction $\mu=0.50$.

Solution.—The student should first note that the coefficient of friction is supposedly known to only two significant figures. (A coefficient of friction is very rarely known any more accurately than this.) Consequently there is no object in taking a value of g more accurate than this, *i. e.*, g should be taken as 32 rather than 32.2 or any more accurate value *without* specific directions.

Let the student also note that unlike Example 1 the equation of motion is not the same for the downward motion as for the upward. There the only force acting along the plane is a component of the weight. Here there is also a force of friction and this *acts downward when the motion is upward and upward when the motion is downward.* Hence the resultant force and the conse-

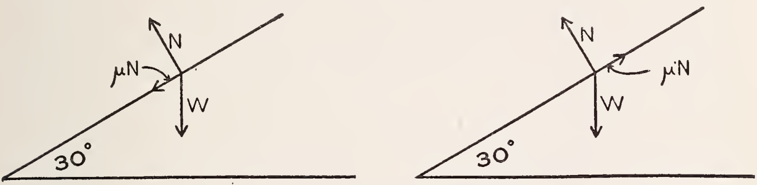


FIG. 198.

quent acceleration are different in the two cases (Fig. 198). For the upward motion

$$F = -W \sin 30° - \mu W \cos 30° = W(-\tfrac{1}{2} - \tfrac{1}{4}\sqrt{3}),$$
$$a = g(-\tfrac{1}{2} - \tfrac{1}{4}\sqrt{3}) = -.93g = -30,$$
$$v = 80 - 30t,$$
$$s = 80t - 15t^2.$$

Now we must observe that we can *not* answer question (1) by putting $t=4$ in this last equation, unless we make sure that this equation of motion continues to hold for 4 seconds. That it actually does not hold when $t=4$ we can readily see by finding v at that instant and observing that it has become negative. This equation holds then only until $v=0$ or $t=2.7$. We can, however, answer question (2) at once by the equation $s = v_0^2/2a = 6400/60 = 107$. The new acceleration is $a_1 = g(-\tfrac{1}{2} + \tfrac{1}{4}\sqrt{3}) = -2$. The time reckoned from the instant of rest is $t_1 = t - 2.7 = 1.3$. Then $s = 107 + \tfrac{1}{2}a_1 t_1^2 = 107 - t_1^2 = 107 - 1.7 = 105$ as accurately as our data warrant. To determine the time to return to the starting point we have $s = 0 = 107 - t_1^2$ or $t_1 = 10.3$, $t = 10.3 + 2.7 = 13$ seconds. Also $v_1^2 = \sqrt{2a_1 s_1} = \sqrt{-4 \times -107} = 21$ f/s. Thus the body takes 10.3 seconds to come down the distance that it went up in 2.7 seconds, and acquires a velocity of only 21 f/s instead of its original velocity of 80 f/s.

Problems

423. A body is projected with a velocity of 80 ft/sec up a smooth inclined plane whose inclination is 30°. Find the space described and the time that elapses before it comes to rest.

424. A particle slides without friction down an inclined plane, and in the fifth second after starting passes over a distance of 2207 cm. Find the inclination of the plane.

425. A particle slides down a rough plane inclined to the horizontal at an angle α. If μ be the coefficient of friction, find the acceleration and velocity when the body has moved a distance l.

426. One body slides down a smooth inclined plane and another drops vertically to the same level. Show that the times are proportional to the spaces described, and that the velocities acquired are equal.

128. Atwood's Machine.—We have been considering so far the motion of a single body. We shall now consider the related motions of two or more bodies which are connected in such a way that when we know the acceleration of one we know the acceleration of the others. The simplest case is that of two bodies directly connected by a weightless, flexible, inextensible cord, which remains taut during the motion, and the part adjacent to each body lies in the rectilinear path of that body. Two weights connected by a cord passing over a pulley, the weights being otherwise unsupported, constitute an Atwood's machine. The pulley is made as light as possible, and otherwise designed to offer as little resistance as possible to motion and acceleration. In a first approximation any resistance of the pulley may be neglected. We may then solve problems connected with an Atwood's machine if we observe the following facts: (1) Newton's Second Law applies to *each body separately* if we use the total resultant force *acting on that body;* (2) The forces acting on each body are its weight and the tension in the cord; (3) Newton's Third Law tells us that

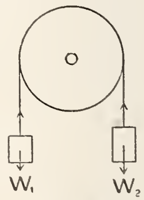


FIG. 199.

the tension in the cord (neglecting the pulley resistance) is the same as applied to each body, but draws one forward and the other backward; (4) the acceleration is the same for the two bodies if that of each is taken in its own path.

Example 1.—The weights on the two sides of an Atwood's machine are 22 pounds and 26 pounds. Find (1) the distance the weights will travel in 3 seconds starting from rest, (2) the tension in the cord, (3) the force necessary to support the pulley during the motion.

Solution.—Applying the principles just enumerated, the equations of motion for the two bodies are $g(26-T)=26a$, and $g(T-22)=22a$, where T is the tension in the cord. Note that the positive direction is the direction of motion in each case, as only in this way do we have the same a in the two equations. In equations of this type T may always be eliminated by direct addition. This gives us $4g=48a=128$. $a=8/3$ f/s². Then $s=\frac{1}{2}at^2=12$. Hence the weights travel 12 feet in the first 3 seconds. Putting the value of a in the original equations enables us to find T and check it. This gives $T=23\frac{5}{6}$ pounds. The pulley is acted on by two downward forces each equal to T. The supporting force therefore is equal to $2T$ or $47\frac{2}{3}$ pounds.

Example 2.—If the weights on an Atwood's machine are 10.000 kilograms and 10.574 kilograms, they move 190.2 cm. in exactly 4 seconds, starting from rest. If one weight is 2.000 kilograms it is found that the other must be 2.176 kilograms to produce the same motion as before. Find the value of g.

Solution.—The two observations enable us to eliminate the resistance of the pulley if we merely assume that the frictional resistance is sensibly constant with the change of load. The angular acceleration of the pulley is actually the same in the two cases. If we take account of the resistance of the pulley, the tensions in the two parts of the cord are different, the difference being the force necessary to produce the motion of the pulley. If we call these in the first trial T_1 and T_1' we have

$$g(10.574-T_1)=10.574a,$$
$$g(T_1'-10.000)=10.000a.$$

Adding these, we have

$$g(0.574-T_1+T_1')=20.574a. \tag{1}$$

If T_2 and T_2' are the tensions in the second trial, we shall have similarly

$$g(0.176-T_2+T_2')=4.176a. \tag{2}$$

But $T_1-T_1'=T_2-T_2'$, as each is the force required to move the pulley. Hence if we subtract (2) from (1) we have $0.398g=16.398a$. But $s=\frac{1}{2}at^2=8a=190.2$, $a=23.78$, $g=16.398\times 23.78\div 0.398=980$, or if the indicated weights are correct to tenths of a gram $g=979.5$ cm/sec.^{2}.

Problems

427. A mass of 9 pounds descending vertically drags up a mass of 6 pounds by means of a string passing over a smooth pulley. Find the acceleration of the mass and the tension of the string.

428. Two equal masses, *m*, are connected by a string passing over a smooth pulley. What mass must be taken from one and added to the other so that each mass may describe 200 feet in 5 seconds?

429. Two masses of 3 pounds each are connected by a light string hanging over a smooth peg. If a third mass of 3 pounds be laid on one of them, by how much is the pressure on the peg increased?

430. A mass of 4 ounces is attached by a string passing over a smooth pulley to a larger mass. Find the magnitude of the latter so that, if the string be cut after the motion has continued for 3 seconds, the former will ascend 16/9 feet before descending.

129. Other Problems on Connected Bodies.—The Atwood's machine is only one case out of many of connected bodies. If two bodies are connected by a cord which lies in their line of motion, the two must have the same acceleration. If the connection is made by means of pulleys, we can easily see the relation between the distances the bodies travel, and hence between their accelerations. If one of the bodies rests on a horizontal or an inclined plane, the pressure of the plane must be taken account of, and

friction if any. Except for pulley resistance, which we shall usually neglect, the tension in a cord is the same throughout.

Example 1.—A weight of 20 pounds rests on a smooth table (Fig. 200). A cord fastened to it passes over a smooth pulley and is attached to a weight of 40 pounds otherwise unsupported. Another cord fastened to the first weight passes over a smooth pulley at the opposite edge of the table and is attached to a weight of 10 pounds which rests on a smooth plane whose slope is $\frac{3}{4}$. Find the acceleration of the system if $g=32.2$.

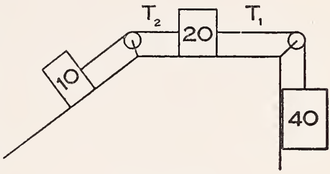


FIG. 200.

Solution.—If T_1 is the tension in the first cord and T_2 that in the second, the equations of motion of the 40, 20, and 10 pound weights are respectively

$$g(40-T_1)=40a,$$
$$g(T_1-T_2)=20a,$$
$$g[T_2-\tfrac{3}{5}(10)]=10a.$$

Adding these three equations, we have

$$g[40-\tfrac{3}{5}(10)]=70a,$$

or

$$a=\frac{34g}{70}=15.7 \text{ f/s}^2.$$

Example 2.—If the coefficient of friction between the 10- and 20-pound weights and their planes in the preceding example is 0.30, find the acceleration of the system.

Solution.—The three equations of motion will now be

$$g(40-T_1)=40a,$$
$$g[T_1-T_2-20(0.30)]=20a,$$

and

$$g[T_2-\tfrac{3}{5}(10)-\tfrac{4}{5}(10)\times(0.30)]=10a.$$

Adding these,

$$g(40-6-6-2.4)=25.6g=70a.$$

Hence

$$a=11.8 \text{ ft/sec}^2.$$

Example 3.—One end of a cord (Fig. 201) is fastened at the top of an inclined plane (inclination 30°). The cord passes over a simple pulley attached to a weight of 100 pounds resting on the plane ($\mu = \frac{1}{2}$) and then passes back over a fixed pulley at the top of the plane and is attached to a weight of 80 pounds otherwise unsupported. Neglecting the pulley resistance, find the acceleration of the 100-pound weight and the tension in the cord.

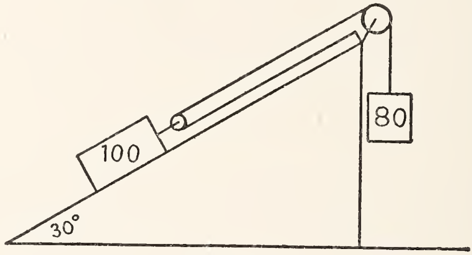


FIG. 201.

Solution.—Let a be the acceleration of the 100-pound weight. Then that of the 80-pound weight will be $2a$. The total force acting to advance the 80-pound weight is $80-T$, where T is the tension in the cord. The forces acting on the 100-pound weight are a tension T in *each* part of the cord, the component of the weight down the plane, 100 sin 30°, and the friction, $\frac{1}{2}(100)$ cos 30°. The equations of motion of the two bodies will then be

$$g(80-T)=80(2a)$$

and

$$g(2T-100 \sin 30° - 50 \cos 30°)=100a.$$

Eliminating T, we have

$$g(160-100 \sin 30° - 50 \cos 30°)=420a,$$

$$a=\frac{110-25\sqrt{3}}{420}g=5.1 \text{ ft/sec}^2.$$

From the first equation,

$$T=80-\frac{160a}{g}=80-\frac{160(110-25\sqrt{3})}{420}=54 \text{ lbs.}$$

Problems

431. A mass of 12 pounds drags a mass of 16 pounds up a smooth plane of inclination 30°, the masses being attached by a string passing over the top of the plane. Find the distance described in 5 seconds and the tension of the string.

432. A mass of 6 ounces slides down a smooth inclined plane whose height is half its length, and draws another mass from rest over a distance of 3 feet in 5 seconds along a smooth horizontal table which is level with the top of the plane over which passes the string which connects the two masses. Find the mass on the table.

433. A mass of 5 pounds on a rough horizontal table is connected by a string to a mass of 8 pounds which hangs over the edge of the table. Find the coefficient of friction in order that the heavier mass may move vertically with half the acceleration it would have if it fell freely.

130. Motion Under a Variable Force.—So far we have discussed only motions in which the force and the resulting acceleration are constant. In many practical problems, however, the force varies. It may depend on the time, on the distance, on the velocity, or on any combination of these. The case of the force depending only on the time is the easiest to handle; but it is of very rare occurrence in practice. It will be included for the sake of completeness. In all problems on vibrations, the force depends on the distance, *i. e.,* on the displacement of the body from its mean position. It depends on the velocity in all problems involving fluid friction or fluid resistance. The possibilities of combination should be obvious. For example, in interior ballistics the force propelling the projectile evidently depends on the time and the distance,—very slightly if at all on the velocity. In exterior ballistics, however, the force retarding the projectile depends only on its velocity, not at all on the time, nor in ordinary cases on the position. The wide range of application is suggested by the few illustrations mentioned. It should also be kept in mind that a vast number of problems in all parts

of Physics are solved in a manner identical with these in Mechanics. In all of these problems we may write $a=\frac{g}{W}F$, where F is a function of t, s, or v, or a combination of them. We may then substitute for a either $\frac{dv}{dt}$, $\frac{vdv}{ds}$, or $\frac{d^2s}{dt^2}$, as may best suit the problem. In nearly all cases g/W will be merely a constant factor. (A rare exception to this last occurs in the case of a moving body such as a rocket which loses part of its substance as it moves. In this case W is variable.)

131. Force Depending on the Time.—Writing $F=F(t)$, we have

$$a=\frac{dv}{dt}=\frac{g}{W}F(t),$$

$$v=\frac{g}{W}\int F(t)dt+C,$$

$$s=\int vdt+C_1.$$

In order to determine C and C_1, we must know two "initial conditions" or the like, *i. e.*, the values of v and s for a particular value of t, or the values of s for two different values of t, or two other similar conditions.

Example 1.—A mass of 8 pounds starts with a velocity of 2 ft/sec and is acted on by a force of $6t^2-4t$ pounds in the direction of its initial velocity, where t is in seconds. Find the distance covered in 3 seconds and the velocity at the end of that interval if $g=32$. Here

$$a=\tfrac{32}{8}(6t^2-4t),$$
$$v=\int a\cdot dt=4(2t^3-2t^2)+C.$$

But $v=2$ when $t=0$. Hence $C=2$ and

$$v=\frac{ds}{dt}=8t^3-8t^2+2.$$

Integrating again,

$$s=2t^4-\tfrac{8}{3}t^3+2t,$$

the constant of integration here being zero, because $s=0$ when $t=0$. By putting $t=3$ in the formulas for s and v, we have that at the end of 3 seconds the body is 96 feet from its starting point and has a forward velocity of 146 ft/sec.

Example 2.—A moving body is observed to pass a fixed point at a certain instant. It is observed again at the ends of 2, 4, and 6 seconds to be 202, 368, and 486 cm. respectively from this point. If it is subjected to a retarding force which is proportional to the time elapsed since the instant the force is first applied, determine (a) the original velocity, (b) when and where the force was first applied, (c) when and where the body comes to rest, (d) the mass of the body in grams if the retarding force when it comes to rest is 250 dynes.

Solution.—Let t_o be the instant when the force is first applied. Then the force and consequently the acceleration will be proportional to $t-t_o$, or

$$a=k(t-t_0). \tag{1}$$

Integrating this twice, we have

$$v=\tfrac{1}{2}k(t-t_0)^2+v_0, \tag{2}$$

and

$$s=\tfrac{1}{6}k(t-t_0)^3+v_0t+C, \tag{3}$$

where v_o is the original velocity when the force is applied. If s is measured from the first observed point, we have for determining the four unknown constants, k, t_0, v_0, and C, the four values of s, namely, 0, 202, 368, and 486, corresponding to $t=0$, 2, 4, 6, respectively. Putting these in (3),

$$0=\tfrac{1}{6}k(-t_o)^3+C; \tag{4}$$
$$202=\tfrac{1}{6}k(2-t_o)^3+2v_o+C; \tag{5}$$
$$368=\tfrac{1}{6}k(4-t_o)^3+4v_o+C; \tag{6}$$
$$486=\tfrac{1}{6}k(6-t_o)^3+6v_o+C. \tag{7}$$

Subtracting (4) from each of the succeeding, and multiplying by 6,

$$1212=k(8-12t_o+6t_o^2)+12v_o; \tag{8}$$
$$2208=k(64-48t_o+12t_o^2)+24v_o; \tag{9}$$
$$2916=k(216-108t_o+18t_o^2)+36v_o. \tag{10}$$

Eliminating v_0 between (8) and each of the others,

$$-216=k(48-24t_0)\,; \tag{11}$$
$$-720=k(192-72t_0). \tag{12}$$

Finally, eliminating t_0,

$$-72=48k, \qquad k=-\tfrac{3}{2}.$$

Substituting successively in (11), (8), and (4), we get

$$t_0=-4,\ v_0=120, \text{ and } C=16.$$

Hence the equation of motion is, from (3),

$$s=-\tfrac{1}{4}(t+4)^3+120t+16.$$

When $t=t_0=-4$, $s=-464$. Hence, answering (b), the force was applied 4 seconds before the first observation, and at a point 464 cm. from the first observed point. Substituting in (2),

$$v=-\tfrac{3}{4}(t+4)^2+120.$$

When the body comes to rest $v=0$, and hence

$$t=\sqrt{160}-4=8.65 \text{ and } s=-\tfrac{1}{4}(160)^{\frac{3}{2}}+120(8.65)+16=548.$$

These results answer (c). Substituting in (1),

$$a=-\tfrac{3}{2}(t+4).$$

When $t=8.65$, this gives $a=-18.97$ and

$$m=\frac{F}{a}=\frac{-250}{-18.97}=13.2 \text{ grams.}$$

132. Force Depending on the Distance.—In this case it is always advisable to express the acceleration in the form

$$a=\frac{dv}{dt}=v\frac{dv}{ds}.$$

We then have a first-order differential equation connecting v and s. If we can integrate this we have an equation (without derivatives) between v and s. Then putting $v=\dfrac{ds}{dt}$ we have to integrate another first-order equation, in s and t, in order to finish the solution of the problem.

We should bear in mind the importance of the correct determination of the constants of integration. This is done by the same methods as those used in the examples of the preceding article.

Example 1.—A body when displaced from its position of equilibrium is acted on by a force tending to restore it, which is proportional to the amount of the displacement. If this force is 40 pounds when the displacement is 2 feet, determine the motion of the body if it weighs 160 pounds and starts from rest with a displacement of 4 feet.

Solution.—If s denotes the distance of the body from its position of equilibrium, then F is proportional to s; that is, $F=ks$. To determine k we have that $F=-40$ when $s=2$, the sign of F being negative because the force tends to decrease s. Hence $k=-20$, and $F=-20s$. Note that this makes F negative when s is positive and positive when s is negative, and thus agrees with the physical fact that the force always tends to restore the body to the position of equilibrium, and hence acts in the direction opposite to the displacement. The equation of motion is then

$$gF=-20gs=160a;$$

$$a=-\tfrac{1}{8}gs=v\frac{dv}{ds};$$

$$2v\,dv=-\tfrac{2}{8}gsds;$$

$$v^2=-\tfrac{1}{8}gs^2+C.$$

To determine C we know that $v=0$ when $s=4$. Hence $C=2g$ and

$$v^2=\frac{g}{8}(16-s^2).$$

Before performing the second integration we may infer several characteristics of the motion from this equation in v and s. Thus, as regards numerical values, v decreases as s increases; becomes zero when s reaches a maximum, and reaches a maximum when $s=0$. Also at any distance s from the mean position the value of v^2 is determined, but v may be positive or negative. This shows that the body passes any given point with the same speed whether going or returning.

If we continue the integration, we have

$$v=\frac{ds}{dt}=\sqrt{\frac{g}{8}}\sqrt{16-s^2};$$

$$t=\sqrt{\frac{8}{g}}\int\frac{ds}{\sqrt{16-s^2}}=\sqrt{\frac{8}{g}}\sin^{-1}\frac{s}{4}+C_1.$$

When $t=0$ $s=4$; hence

$$0=\sqrt{\frac{8}{g}}\cdot\frac{\pi}{2}+C_1$$

and

$$t=\sqrt{\frac{8}{g}}\left(\sin^{-1}\frac{s}{4}-\frac{\pi}{2}\right)=\sqrt{\frac{8}{g}}\cos^{-1}\frac{s}{4}.$$

Solving for s, we have

$$s=4\cos\sqrt{\frac{g}{8}}\,t.$$

This shows us, in addition to the information we had before, that the motion is *periodic;* that is, it exactly repeats itself after a certain time interval. For the trigonometric function $\cos\sqrt{\frac{g}{8}}\,t$ will return to the same value if $\sqrt{\frac{g}{8}}\,t$ is increased by 2π; that is, if t is increased by $2\pi\sqrt{\frac{8}{g}}$. This quantity is called the *period* of the motion. The maximum displacement, in this case 4 feet, is called the *amplitude.*

133. Simple Harmonic Motion.—The preceding example is an illustration of *Simple Harmonic Motion.* This is defined in general as motion in which the force is proportional to the displacement of the body from its mean position and acts in a direction to restore it to that position. From the equation of motion it follows that the acceleration is equal to a negative constant times the displacement. If this constant is called $-k^2$, we have $a=-k^2s$. A complete solution of this differential equation is

$$s=\frac{v_0}{k}\sin k(t-t_0),$$

where t_0 and v_0 are the time and the velocity when $s=0$. The period of this motion is $2\pi/k$ and its amplitude is v_0/k. The quantity t_0, which determines when the motion starts, is called the *phase*. Frequently instead of the phase, the *phase angle* is spoken of. This is kt_0 and is often given in degrees instead of radians, especially in the treatment of alternating currents, which is very closely related to this subject in Mechanics.

Problems

434. Integrate the equation $a=-k^2s$ and obtain the general equation of simple harmonic motion with the constants expressed in terms of t_0 and a, the phase and amplitude. (The method of solution is precisely the same as in the numerical example which was worked out.)

435. Show that the period of a simple harmonic motion depends only on the force of restitution and not at all on the initial conditions.

436. Express the maximum speed in terms of the amplitude.

437. Show that in any simple harmonic motion the acceleration and displacement reach numerical maxima when the velocity is zero, and are equal to zero when the velocity is a maximum.

438. Find the period, amplitude, phase angle, and maximum velocity for the motion

$$s=3 \sin (4t-\pi).$$

439. By expanding $\sin k(t-t_0)$ as the sine of a sum show that any simple harmonic motion can be written in the form

$$s=A \sin kt+B \cos kt,$$

and express A and B in terms of amplitude, period and phase.

440. A certain spring is stretched one inch by a force of 10 pounds. A weight of 20 pounds is hung on the spring in its unstretched condition. Assuming the spring obeys Hooke's Law, determine the amplitude and period of the resulting motion.

441. Write the equation of a simple harmonic motion having the amplitude 5 feet, period 3 seconds, and phase angle 60°.

442. Below the surface of the earth the force of gravity is directly proportional to the distance from the center. If a hole could be bored straight through the earth, and a body were

dropped in, how long would it take to reach the antipodes and what would be its maximum velocity?

134. Force Depending on the Velocity.—*Example.*—A body whose motion is retarded by a fluid friction has an acceleration $-kv$ where v is the velocity. Find the distance covered in time t if the initial velocity is v_0.

Solution.—

$$a = -kv = \frac{dv}{dt},$$

$$-k\,dt = \frac{dv}{v},$$

$$-kt = \log v + C = \log v - \log v_0 = \log \frac{v}{v_0};$$

$$\therefore\ v = v_0 e^{-kt},$$

$$s = -\frac{v_0}{k} e^{-kt} + C = \frac{v_0}{k}(1 - e^{-kt}),$$

if $s = 0$ when $t = 0$.

Problems

443. Show that the distance covered in the preceding example can never exceed v_0/k.

444. Solve the same problem if the retarding force is proportional to the square of the velocity.

445. In this last problem is there (theoretically) any upper limit to the distance covered?

Review Problems

446. A 10-pound body moving 4 ft/sec meets a 12-pound body moving 7 ft/sec in the opposite direction. If they unite into one body, find its velocity.

447. An Atwood's machine, with suspended masses m_1 and m_2, is placed on the platform of a set of scales. Find the change in its apparent weight when the masses are allowed to move.

448. A string hung over a pulley has at one end a mass of 10 pounds and at the other end two masses of 8 and 4 pounds respectively. After being in motion for 5 seconds the 4-pound mass is taken off. How much further will the masses go before they come to rest?

449. A train of mass 200 tons is running at the rate of 40 mi/hr down an incline of 1 in 120. Find the resistance necessary to stop it in half a mile.

450. A car starting from rest coasts for 1 mile down an incline of 1 in 100. If the resistance be equal to 8 pounds per ton, how far will the car be carried along the horizontal level at the foot of the incline?

451. A stone is dropped into a well and the sound of the splash is heard in $7\frac{7}{10}$ seconds. If the velocity of sound is 1120 ft/sec how deep is the well?

452. A cage in a mine shaft descends with 2 ft/sec^2 acceleration. After it has been in motion for 10 seconds, a particle is dropped on it from the top of the shaft. When will the particle hit the cage?

453. A particle of unit mass moves in a straight line so that $s = 6 - 5 \sin^2 \frac{\pi t}{2}$, where t is the time and s the distance from a point O. Find when the particle is moving forward and when backward. Find also the greatest distance which the particle reaches from O and the force which acts upon it.

454. A particle falls from height h upon a horizontal plane. If e is the coefficient of restitution, find the whole distance described by the particle, and the time that elapses before it has finished rebounding.

455. The masses of 5 balls at rest in a straight line form a geometric progression whose ratio is 2 and their coefficients of restitution are each $\frac{2}{3}$. If the first ball be started toward the second with velocity v, find the velocity communicated to the fifth.

456. Two scale pans, each of mass 3 pounds, are connected by a string passing over a smooth pulley. Show how to divide a mass of 12 pounds between the two pans so that the heavier may descend a distance of 50 feet in the first five seconds.

457. Two strings pass over a smooth pulley; on one side they are attached to masses of 3 and 4 pounds respectively and on the other both are attached to a single mass of 5 pounds. Find the tensions of the strings and the acceleration of the system.

458. A mass m pulls a mass m' up a smooth plane of inclination β, by means of a string passing over a pulley at the top of the plane. Find the acceleration of the system.

CHAPTER IX

WORK AND ENERGY

135. Work.—When a force acts on a fixed point of a body, and the point of application moves so that the displacement, however small, has a component in the direction of the force, the latter is said to do work. Thus a force does work on a particle when and only when the particle moves under the action of the force. Force and distance are essentials of work. Force, however long it may act, which produces no motion, does no work. For example, a weight of 100 pounds rests on a rough horizontal plane for which $\mu=\frac{1}{4}$. A horizontal force of 15 pounds is applied. The body will not move, since the force of friction (which may be as great as 25 pounds) will be just sufficient to produce equilibrium. According to the mathematical definition of work, the 15-pound force will do no work, however long it acts. This also illustrates that time is not an essential element of work, as mathematically defined. Furthermore, no work results from motion without force. For example, the resultant work done on a particle moving with uniform velocity in a straight line is zero. An unbalanced force always does work, for it always produces motion.

136. Work Done by a Constant Force.—If a constant force F, acting on a particle, displaces it a distance s along the line of action of the force, the work done on the particle by the force is defined by the equation

$$\text{Work}=F\times s.$$

That is, the work is the product of the force and the displacement of the particle.

It appears then that the unit of work involves the unit of force and the unit of length. For example, if a force of A

pounds acts through a distance of B feet or a force of B pounds acts through a distance of A feet, the work done is AB foot-pounds. The work is positive when the displacement is in the direction of the force, and negative when the displacement is in the direction opposite to that of the force. In the former case work is said to be done by the force, while in the latter work is said to be done against the force. Thus, the work done by the force of gravity on a body weighing 20 pounds, which falls 10 feet, is 200 foot-pounds. The work done against the force of gravity by a force which lifts a 20-pound body 10 feet is 200 foot-pounds. The latter may also be expressed by saying the work done by the force of gravity is -200 foot-pounds.

Example 1.—A body of weight 32.2 pounds moves in a straight line with uniformly accelerated motion. If the acceleration is 10 feet per second per second, find the work done on the particle in moving it a distance of 25 feet.

Solution.—From Newton's second law of motion,

$$F = Ma = \frac{32.2}{32.2} 10 = 10 \text{ lbs.}$$

From the definition of work we have,

$$\text{Work} = \text{force} \times \text{displacement} = 250 \text{ ft.-lbs.}$$

Example 2.—A train weighing 500 tons is moving along a straight level track when the power is suddenly shut off. It is then brought to rest in 10 seconds by the friction of the brakes. If the friction is 10 pounds per ton, find the work done by the friction.

Solution.—From Newton's second law of motion,

$$500 \times 10 = \frac{500 \times 2240}{32.2} \cdot a,$$

whence

$$a = \frac{32.2}{224} \text{ ft. per sec. per sec.}$$

The distance the train moves before coming to rest is then

$$s=\tfrac{1}{2}at^2=\frac{1}{2}\,\frac{32.2}{224}\,(10)^2=\frac{1610}{224}\text{ ft.}$$

From the definition of work we have, then,

Work = force times displacement = 35,900 ft.-lbs.

Problems

459. A body weighing 100 pounds rests on a rough horizontal plane ($\mu=0.5$). (a) What work is done in moving the body 10 feet in any direction along the plane? (b) Has the time it takes to move the body any effect upon the work done?

460. A man weighing 140 pounds carries a load of 50 pounds up a ladder 40 feet long inclined at an angle of 60° with the horizontal; how much work does he do?

461. How much work is required to up-end a telephone pole 35 feet long and 1 foot in diameter, if it weighs 50 pounds per cubic foot?

462. A circular well 4 feet in diameter is dug 20 feet deep. Find the work done in raising the material to the surface if the material weighs 140 pounds per cubic foot.

463. Find the work done in raising to the surface the water from a full cistern which is 10 feet square and 6 feet deep.

464. Find how many units of work are stored up in a mill pond 100 feet long, 50 feet broad, and 3 feet deep, the point where the water is discharged being 11 feet below the surface of the pond.

465. An iron plate $\frac{1}{2}$ inch thick has the form of an isosceles trapezoid whose altitude is 6 feet, and whose bases are 15 feet and 5 feet. This iron plate rests on a level pavement. The shorter base is lifted until it is 4 feet above the pavement, the longer base remaining where it was. Find the work done in foot-pounds if the iron weighs 450 pounds per cubic foot.

137. If a constant force F, acting on a particle, displaces it a distance s along a line which makes an angle θ with the line of action of the force, the work done on the particle is defined by the equation

$$\text{Work}=F\cos\theta\times s;$$

that is, the work is the product of the component of the force, in the direction of motion, and the displacement of the particle.

If the component of the force in the direction of motion is called the effective force, the work done is the product of the effective force and the displacement of the particle. The work is positive when the effective force is positive, and is negative when the effective force is negative.

The work done on a particle by the simultaneous action of any number of forces, constant in magnitude and direction, is the work done on it by the component, along the line of motion of the particle, of the resultant of all the forces which act on it.

Example 1.—A block weighing 100 pounds is dragged, at a uniform speed, 100 feet along a rough horizontal plane by a

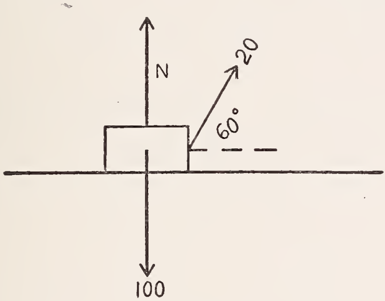


FIG. 202.

force of 20 pounds, whose line of action makes an angle of 60° with the plane. Find the work done on the block by the force.

Solution.—(See Fig. 202.)

$$\text{Effective force} = 20 \times \cos 60° = 10 \text{ lbs.}$$
$$\text{Therefore, work} = \text{effective force} \times \text{displacement}$$
$$= 1000 \text{ ft.-lbs.}$$

Example 2.—A body weighing 67 pounds is drawn, at a uniform speed, 85 feet up a rough plane inclined at arctan $\frac{8}{15}$ to the

horizontal, the drawing force being horizontal. If the coefficient of friction is $\frac{1}{5}$, find the work done:

(a) by the force on the body,
(b) against the force of gravity,
(c) against the force of friction.

Solution.—By Newton's second law of motion, the resultant

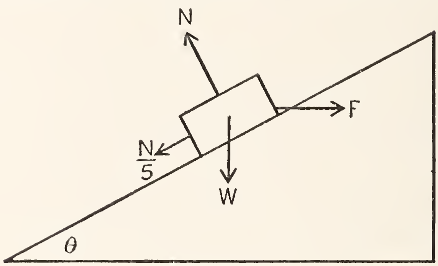


FIG. 203.

force up the plane is zero. Therefore, resolving forces perpendicular to and along the plane (Fig. 203), we find

$$F \sin \theta = N - 67 \cos \theta, \tag{1}$$

and

$$F \cos \theta = N/5 + 67 \sin \theta. \tag{2}$$

From (1) and (2) we get

$$N = 85 \text{ lbs. and } F \cos \theta = 48\tfrac{9}{17} \text{ lbs.}$$

(a) Work = effective force × distance = $48\frac{9}{17} \times 85 = 4125$ ft.-lbs.

(b) Work = effective force × distance =

$$(67 \times \tfrac{8}{17}) \times 85 = 2680 \text{ ft.-lbs.}$$

(c) Work = effective force × distance = $\frac{85}{5} \times 85 = 1445$ ft.-lbs.

Problems

466. A weight of 20 pounds is dragged 50 feet up a plane inclined at 30° to the horizontal by a constant force $P = 25$ pounds, acting at an angle of 15° to the plane. There is a retarding force $R = 5$ pounds along the plane. Compute the work done by or against each force.

467. The frictional resistance along an incline of 1 in 25 being 150 pounds per ton, how much work is done when 2 tons are dragged at a uniform speed 100 feet along the plane?

468. The frictional resistance along an incline of 1 in 30 being 180 pounds per ton, how much work is done when 3 tons are dragged at a uniform speed 90 feet along the plane?

469. A weight of 26 pounds is moved 13 feet up a rough plane ($\mu=\frac{1}{3}$) inclined at an angle of arctan $\frac{5}{12}$ to the horizon, by a force of 20 pounds acting at an angle of arctan $\frac{3}{4}$ to the inclined plane. Find the work done against gravity, and against friction.

470. A weight of 39 pounds is moved 13 feet up a rough plane, ($\mu=\frac{1}{4}$), whose base is 12 feet and whose height is 5 feet, by a constant force of 50 pounds acting parallel to the plane. Find (a) the work done against friction, and (b) the work done against gravity.

471. Find the work done if a body weighing w pounds is dragged up a rough plane whose length is l feet, base b feet, and height h feet. The coefficient of friction is μ, and the velocity is constant.

138. Energy.—By the energy of a particle is meant the work which the particle possessing it is capable of doing. It appears then that the unit of energy is the same as the unit of work.

If a stone weighing 5 pounds falls freely through a distance of 100 feet the force of gravity does 500 foot-pounds of work. It is thus seen that the stone, when at the elevation of 100 feet, has 500 foot-pounds of energy stored up in it. The energy which a particle possesses by virtue of its position of advantage is called potential energy. Thus a particle of weight W at a height H has $W \times H$ units of work stored up in it, or possesses $W \times H$ units of potential energy.

It is a matter of experience that every moving particle can do work upon another particle. It follows then that every moving particle possesses energy. The energy which a particle possesses by virtue of its motion is called kinetic energy.

The total energy of a body is the sum of its kinetic and potential energies. In most systems which we shall consider, this sum remains constant. Such systems are called conservative.

139. Expression for Kinetic Energy.—Suppose a particle of mass M, moving in a straight line with a velocity v, is brought to rest in a space s by the action of a constant force whose component along the line of motion is F. From Newton's second law of motion,

$$-F=Ma=M\cdot\frac{dv}{dt}=M\cdot\frac{ds}{dt}\cdot\frac{dv}{ds}=Mv\cdot\frac{dv}{ds},$$

the minus sign being used because the force F tends to decrease the velocity. It follows then that

$$\int_0^s F\,ds=-\int_v^0 Mv\,dv,$$

or

$$F\int_0^s ds=M\int_0^v v\,dv.$$

Therefore,

$$Fs=\tfrac{1}{2}Mv^2.$$

But Fs is the work the particle can do in consequence of its motion. Hence the kinetic energy of a particle of mass M moving in a straight line with a velocity v is $\frac{1}{2}Mv^2$. For example, the kinetic energy of a body of weight 64.4 pounds moving with a velocity of 25 feet per second is

$$\tfrac{1}{2}M\,v^2=\tfrac{1}{2}\cdot\frac{64.4}{32.2}(25)^2=625 \text{ ft.-lbs.}$$

Problems

472. A body of weight 64.4 pounds has its velocity reduced from 88 feet per second to 44 feet per second. Find the change in its kinetic energy.

473. A body weighing 30 pounds slides down a rough inclined plane ($\mu=\frac{1}{5}$) of height 30 feet and base 100 feet. Find the kinetic energy acquired.

474. A body weighing 39 pounds is moved up a rough inclined plane ($\mu = 0.25$) of height 5 feet and base 12 feet, by a constant force of 50 pounds, acting parallel to the plane. Find the kinetic energy acquired by the body when it is 10 feet from the starting point.

475. What is the kinetic energy of a car weighing 25 tons moving 6 miles per hour and loaded with 36 passengers of average weight 154 pounds?

476. A weight of W pounds is drawn up a smooth plane, inclined 30 degrees to the horizontal, by an equal weight which falls vertically: the weights are connected by a string which passes smoothly over the top of the plane. If the first weight is initially at rest and at a distance of 16 feet from the top of the plane, find the combined kinetic energy of the weights, just as the first one is about to slip off the plane.

140. Work Done by a Force, Constant in Direction but Variable in Magnitude.—Suppose a variable force, acting on a particle, displaces it from P_0 to P (Fig. 204), along the line of

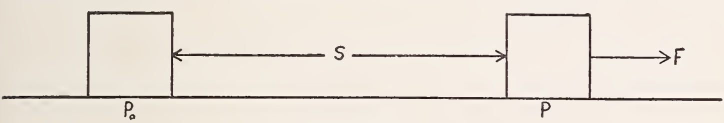


FIG. 204.

action of the force. Let P be the position of the particle at any time and denote by s the distance from P_0 to P. In this consideration F is assumed to be a function of s. We now seek a definition of the work to be done on the particle by the force.

Let the interval from P_0 to P be divided into n equal parts and designate by Δs the common length of these sub-intervals. Denote by F_i the value of F at the beginning of the ith interval. If we assume that F_i remains a constant throughout the ith interval, the work done on the particle by F would be

$$F_1\Delta s + F_2\Delta s + F_3\Delta s + \; .\; .\; .\; . \; + F_n\Delta s.$$

The limit of this sum, when n increases without limit, is the integral of Fds between the limits 0 and s, and we are led to define the work done in this case by the equation

$$\text{Work} = \int_0^s Fds.$$

If s is measured from a point on P_oP, such that s_o is the coordinate of P_o and s the coordinate of P, then

$$\text{Work} = \int_{s_0}^s Fds.$$

In finding the work done in any particular case F is expressed as a function of s and then Fds is integrated between the proper limits.

Example 1.—A weight of 400 pounds compresses an elastic spring 1 inch. Find the work done in compressing it 6 inches.

Solution.—By Hooke's law

$$F = ks.$$

Since $s=1$ when $F=400$,

$$k=400 \text{ and } F=ks=400s \text{ lbs.}$$

Therefore

$$\text{Work} = \int_{s_0}^s Fds = \int_0^6 400sds = 7200 \text{ in.-lbs.}$$

Example 2.—Find the work done on a stone of weight 10 pounds which falls from interstellar space to the earth's surface under the action of the force which the earth alone exerts.

Solution.—Denote by R the radius of the earth and by s the distance of the stone from the earth's center at any instant. Then

$$F = \frac{k}{s^2}.$$

Since $F=-10$ when $s=R$,

$$-10 = k/R^2 \text{ and } F = -10R^2/s^2.$$

Therefore

$$\text{Work} = \int_\infty^R Fds = \int_\infty^R -10R^2 \frac{ds}{s^2} = 10R^2\int_R^\infty \frac{ds}{s^2}$$
$$= 10R = 211,200,000 \text{ ft.-lbs.}$$

Example 3.—An engine which uses compressed air has a cylinder of length 24 inches and a cross-sectional area of 16 square inches. Air at a pressure of 100 pounds per square inch is admitted into the cylinder during the first half of the stroke. The valve is then closed. Find the work done on the piston during the forward stroke, assuming the expansion of the air obeys the law

$$pv^{1.408} = \text{ a constant,}$$

p being the intensity of pressure and v the volume of the air at any instant.

Solution.—(See Fig. 205.) The working force for the first

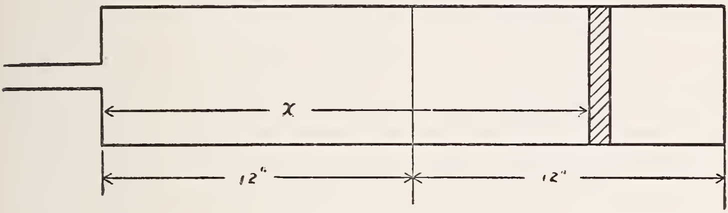


FIG. 205.

half of the stroke is

$$F_1 = \text{pressure} \times \text{area} = 100 \times 16 \text{ pounds, a constant force.}$$

The working force for the second half of the stroke is

$$F_2 = \text{pressure} \times \text{area} = p \times 16 \text{ pounds, a variable force.}$$

Since $p = 100$ when $v = 12 \times 16$, we get

$$k = 100(12 \times 16)^{1.408}$$

and

$$p = \frac{k}{v^{1.408}} = 100 \left(\frac{12 \times 16}{v}\right)^{1.408} = 100 \left(\frac{12}{x}\right)^{1.408}.$$

Therefore the working force for the second half of the stroke is

$$F_2 = p \times 16 = 1600 \left(\frac{12}{x}\right)^{1.408} \text{ lbs.}$$

The work done is then

$$12\times F_1+\int_{12}^{24}F_2dx=12(1600)+1600(12)^{1.408}\int_{12}^{24}\frac{dx}{x^{1.408}},$$
$$=19200+\frac{1600\times 12}{0.408}\left[1-(0.5)^{0.408}\right],$$
$$=30790 \text{ in.-lbs.}=2566 \text{ ft.-lbs.}$$

141. If the force F makes an angle θ with the direction of motion the work is defined by the equation

$$\text{Work}=\int_{s_0}^{s}F\cos\theta\, ds.$$

That is, the work is equal to that done by the effective force.

In case a number of forces, constant in direction and variable in magnitude, act on the particle simultaneously, the work done by them on the particle is the same as that done by the component, along the line of motion of the particle, of the resultant of all the forces which act on it.

Problems

477. A weight of 300 pounds hangs by a chain 100 feet long which weighs 2.5 pounds per foot. How much work is required to lift the weight by winding up the chain on a windlass?

478. A spring is compressed from 18 inches to 14 inches by a weight of 504 pounds. How much work must be done to compress it two more inches?

479. A weight of 500 pounds hangs by a chain 500 feet long which weighs 2 pounds per foot. How much work is necessary to lift the weight by winding up the chain on a windlass?

480. It requires a force of 960 pounds to compress the spring of a safety valve from 20 inches to 17 inches. How many foot-pounds of work must be done to compress it 1 inch further?

481. A steam engine has a cylinder 18 inches long with a diameter of 12 inches. Steam with boiler pressure of 120 pounds per square inch is admitted into the cylinder during the first third of the stroke. The port is then closed, and assuming that

the pressure due to the expansion of the steam follows Boyle's law (pv=a constant), find the work done on the piston during the forward stroke.

482. A spring is normally 21 inches long; a force of 5 pounds compresses it 4 inches; how much work would be done in compressing it so that its final length is 12 inches?

142. Power.—A force, however small, can do a work of any magnitude provided sufficient time is allowed. The work required to lift 100 tons of coal from a mine 180 feet deep is the same whether the task is performed in one hour or in eight hours. Since in practice time is a most important factor, it is necessary, in comparing agents which do work, to consider the time employed. The rate at which work is done is called *power*. Hence the unit of power is the unit of work divided by the unit of time. The horsepower (H. P.) is the unit of power commonly used in engineering practice, and may be defined as the power of an agent which can do 550 foot-pounds of work per second, or 33,000 foot-pounds of work per minute. For example, to hoist 3300 pounds of mortar to the top of a structure 150 feet high in 5 minutes would require 99,000 foot-pounds of work per minute, or 1650 foot-pounds of work per second; therefore an agent which can deliver 3 H. P. is needed.

Example 1.—What horsepower is necessary to keep a 150-ton train moving on a level track at 40 miles per hour, the frictional resistance being 20 pounds per ton?

Solution.—Since there is no acceleration, the force exerted on the train must equal the total resistance.

$$F = 20 \times 150 = 3000 \text{ lbs.}$$

By definition,

$$\text{H.P.} = \frac{Fs}{550 \cdot t} = \frac{Fv}{550} = \frac{3000 \times 176/3}{550} = 320.$$

Example 2.—A steel rod 12 square inches in cross-sectional area and 6 feet long is to be stretched 1100 times per minute. Find the horsepower required to produce in it a tension of 72,000 pounds. ($E = 3 \times 10^7$ lbs./in.^{2}. See Arts. 85-87.)

Solution. — Work per stretch = average force × distance stretched;

$$W=\frac{F}{2}\cdot x=\frac{F}{2}\cdot\frac{lp}{E}=\frac{72000}{2}\cdot\frac{72\left(\frac{72000}{12}\right)}{3\times 10^7}=518.4\text{ in.-lbs.}$$

$$\text{H. P.}=\frac{518.4\times 1100}{12\times 33000}=1.44.$$

Problems

483. How many pounds per hour will a 10 H. P. engine hoist from a depth of 600 feet, supposing 20 per cent of the power is lost in friction?

484. What H. P. is required to keep a 200-ton train moving on a level track at 40 miles per hour, if the frictional resistance is 20 pounds per ton?

485. A 600-ton train is moving at 15 miles per hour up a grade of 1 in 200. If the frictional resistance is 10 pounds per ton, what horsepower is the engine developing?

486. A motor car weighing 0.75 tons runs (without power) down a grade of 1 in 30 at 12 miles per hour (no acceleration). What horsepower must it have to run up the same grade at the same speed?

487. What horsepower is necessary to keep a 100-ton train moving on a level track at 30 miles per hour, the frictional resistance being 20 pounds per ton?

488. A train weighing 200 tons is moving up a slope of 1 in 85 at 20 miles per hour. If the frictional resistances are 12 pounds per ton, what horsepower is the engine developing?

489. A train weighing 80 tons is climbing a 2 per cent grade, the frictional resistances being 10 pounds per ton. If the engine is developing 440 horsepower at the instant the speed is 30 miles per hour, what is the acceleration at that instant?

490. An engine pulling a certain load exerts a constant force of 500 pounds and travels at the rate of 20 miles per hour. (a) How much work does it do in 5 minutes? (b) What horsepower is it developing?

491. A 6-inch rapid-fire gun discharge 5 projectiles per minute, each weighing 100 pounds, with a velocity of 2800 feet per second. (a) What kinetic energy does each projectile possess on leaving the gun? (b) What horsepower is the gun developing?

143. Relation Between Kinetic Energy and Work.—Suppose a particle of mass M moves from P_o to P_1 under the action of any number of forces, constant in direction, and denote by F the component, along the line of motion of the particle, of the resultant of all the forces which act on it. Let v_o be the velocity of the particle at P_o and v_1 its velocity at P_1.

From Newton's Second Law of Motion

$$F=Ma=M\frac{dv}{dt}=M\frac{ds}{dt}\frac{dv}{ds}=Mv\frac{dv}{ds}.$$

Then

$$\int_{P_0}^{P_1} F ds=\int_{v_0}^{v_1} Mv\ dv=\tfrac{1}{2}Mv_1^2-\tfrac{1}{2}Mv_o^2.$$

That is, the change in the kinetic energy of a particle equals the work done on it by the component, along the line of motion of the particle, of the resultant of all the forces which act on it.

Example 1.—Find the work done on a body of weight 64.4 pounds by a force that changes its velocity from 10 feet per second to 20 feet per second in a distance of 40 feet against a uniform resistance of 25 pounds.

Solution.—The force must overcome the resistance and also produce the change in velocity.

The work required to overcome the resistance is

$$\text{Work}=\text{force}\times\text{distance}=25\times 40 \text{ ft.-lbs.}=1000 \text{ ft.-lbs.}$$

The work required to produce the change in velocity is

$$\text{Work}=\tfrac{1}{2}M(v_1^2-v_o^2)=\tfrac{1}{2}\,\frac{64.4}{32.2}\,(400-100) \text{ ft.-lbs.}=300 \text{ ft.-lbs.}$$

The total work done by the force is then

$$1000 \text{ ft.-lbs.}+300 \text{ ft. lbs.}=1300 \text{ ft.-lbs.}$$

Example 2.—A weight W is projected down the rough incline AB ($\mu=0.25$) from B with an initial velocity of 10 feet per

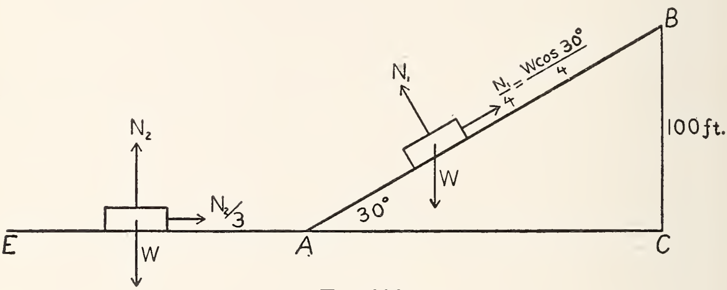


FIG. 206.

second. Find the distance W will travel on the rough horizontal plane AE ($\mu=\frac{1}{3}$) before coming to rest. (Assume no shock at A.) (See Fig. 206.)

Solution.—The kinetic energy of W at B is

$$\tfrac{1}{2}Mv^2=\tfrac{1}{2}\frac{W}{g}v^2=\frac{50}{32.2}W.$$

The kinetic energy acquired by W in sliding from B to A is the work done on the particle:

$$(W\sin 30°-\frac{W}{4}\cos 30°)\times 200.$$

Hence the kinetic energy of W at A is

$$\frac{50W}{32.2}+(W\sin 30°-\frac{W}{4}\cos 30°)\times 200=\frac{50W}{32.2}+100W-25\sqrt{3}W.$$

The body will move along AE until its kinetic energy is overcome by the resistance $W/3$.

Hence, if S denotes the distance W travels along AE, we have:

$$\frac{W}{3}\times S=\frac{50W}{32.2}+100W-25\sqrt{3}W$$

and

$$S=174.8 \text{ ft.}$$

Problems

492. A car weighing $26\frac{11}{14}$ tons breaks loose on a level track from a train moving 15 miles per hour. The frictional resistance (constant) is 300 pounds. How long before the car will stop, and how far will it have gone from where it broke loose?

493. A weight of 20 pounds is projected along a rough horizontal plane with an initial velocity of 8 feet per second. If the frictional resistance is 4 pounds, how far will the weight move before coming to rest?

494. A 30-ton car is moving with a velocity of 30 miles per hour on a level track. The brakes refuse to work. How far

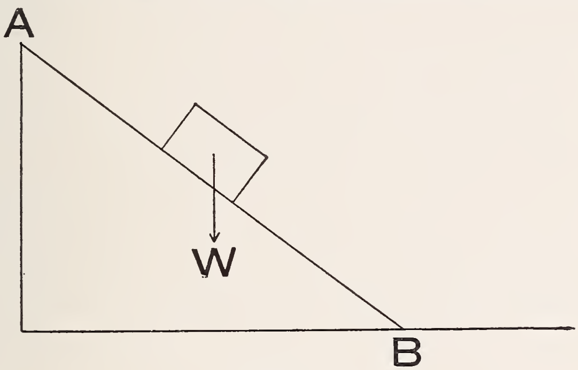


FIG. 207.

will the car go after the power is turned off before coming to rest if the friction is 1 per cent of the weight of the car?

495. An automobile going at the speed of 30 miles per hour comes to the foot of a hill. The power is then shut off. The hill rises 1 in 50. How far up the hill will the automobile go if the friction is 6 per cent of the weight of the car?

496. A body of weight W starts at A and slides down a rough plane ($\mu = 0.25$) 100 feet long and inclined at an angle of arctan 0.75 to the horizontal. At the lowest point, B, of the plane the body runs on to a rough horizontal plane ($\mu = 0.125$). Find the distance from B the body comes to rest. (Assume no loss of kinetic energy at B.) (See Fig. 207.)

497. Solve last problem if the body has at A an initial velocity of 32.2 feet per second.

Review Problems

498. A vat which is a right prism whose cross section is an isosceles triangle is full of fresh water. The length of the vat is 20 feet, its depth is 9 feet, and the distance across its top is 12 feet. Find the work done in pumping the water out of the vat, if it is to be discharged out of an orifice 8 feet above the top of the vat. A cubic foot of fresh water weighs 62.5 pounds.

499. Find the work done in emptying a right circular cone of height h and radius of base a of a liquid which weighs w pounds per cubic foot. The axis of the cone is vertical and its vertex is down.

500. A rectangular tank, 3 feet long, 2 feet wide, and 1.5 feet deep is filled from a cylindrical tank of 24 square feet horizontal cross section, in which the water level at the start is 20 feet below the bottom of the rectangular tank. How much work is done, assuming that the intake pipe enters the bottom of the rectangular tank?

501. An engine of 10 horsepower raises 4000 pounds of coal in an hour from a pit 1200 feet deep, and also gives motion to a hammer which makes 50 lifts of 5 feet each in a minute. Find the weight of the hammer, assuming the efficiency of the engine is unity.

502. A chain weighs 10 pounds per foot and is 300 feet long. It hangs from a drum into a mine shaft. How much work is done in winding up 100 feet of it?

503. A weight of 300 pounds hangs by a chain 100 feet long. The chain weighs 2.5 pounds per foot. How much work must be done to lift the weight 50 feet by winding the chain upon a windlass?

504. What horsepower is necessary to keep a 150-ton train moving on a level track at 40 miles per hour, the frictional resistance being 20 pounds per ton?

505. A 2-ton car runs down a slope of 1 in 20 at 10 miles per hour, the resistance at that speed being just sufficient to prevent acceleration. What horsepower must it exert to run up the slope at the same speed?

506. What is the effective horsepower of a locomotive which gives a 500-ton train a speed of 15 miles per hour up a grade of 1 in 320, the frictional resistance being 12 pounds per ton?

507. A car weighing 3000 pounds runs down a slope of 1 in 20 at 15 miles per hour, the resistance at that speed being just sufficient to prevent acceleration. What horsepower must be exerted to make the car run up the slope at the same speed?

508. A 400-ton train is moving up a grade of 1 in 100 against a frictional resistance of 10 pounds per ton. What horsepower will the engine develop at the instant when the speed is 15 miles per hour, if the acceleration of the train at that instant is 0.2 feet per second per second?

509. An 8-ton trolley car starts at A with a speed of 15 miles per hour, and reaches B with a speed of 22.5 miles per hour: if

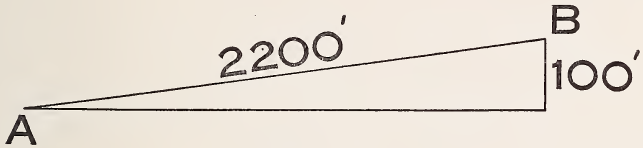


FIG. 208.

the average frictional resistance is 15 pounds per ton, and the driving force remains constant, what horsepower is being developed at B? (Fig. 208.)

510. What is the horsepower of an engine that can raise every minute and a half 500 cubic feet of water to a height of 100 feet?

511. Find the horsepower of an engine which is drawing 120 tons up an incline of 1 in 300 at 30 miles per hour against a frictional resistance of 20 pounds per ton.

512. A slider weighing 100 pounds rests on a table. It is moved by a weight of 20 pounds fastened to it by a rope which passes over a pulley at the edge of the table. When the slider has moved 2 feet its velocity is 2 feet per second. Find the coefficient of friction.

513. In the motion of a particle down a smooth cycloid the vertical component of its velocity is greatest when it has completed what fraction of its vertical descent?

514. The average pressure on the piston of a steam engine is 60 pounds per square inch, the area of the piston is 1 square foot, the length of the stroke is 18 inches, the pressure occurs only in the forward motion. How many strokes does the engine make per minute if it registers 8 horsepower?

515. Unit particles, which attract each other according to the law of the inverse square, are situated at A and B. The particle at B is held fast and the particle at A is constrained to move in the arc of the circle. Find the amount of work done by the attracting force when the particle at A has moved one-third the way to B. (Fig. 209.)

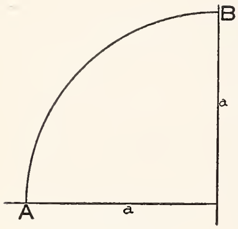


FIG. 209.

516. A particle of mass M is fixed at the origin and attracts according to the law of the inverse square a particle of mass m which is constrained to move in the line $y=3$. Show that the work done by the attractive force in moving the particle from the point where $x=4a$ to the y-axis is $4Mm/5a$.

517. Find by integration the work done against gravity in bringing a weight w from A to B along the circle. (See Fig. 209.)

CHAPTER X

CURVILINEAR MOTION

144. Vector Velocity.—When we considered motion along a straight line, the location of a moving particle at any instant could be given by a scalar quantity, namely, its distance from a fixed point on the line. We shall now consider the motion of a particle in a curve. To determine its position we must now have given a *vector,* namely, that from some fixed point O as origin to the position of the particle. This is called the position vector of the moving particle. As the particle moves, this vector changes in

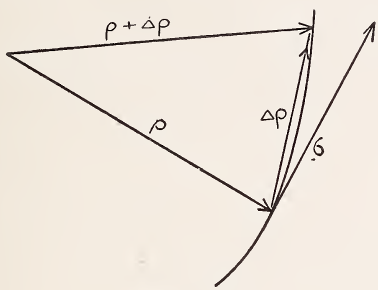


FIG. 210.

direction or magnitude or both. Let ρ and $\rho + \Delta\rho$ (Fig. 210) be the position vectors of a particle at the beginning and at the end of a time interval Δt. Then the vector difference $\Delta\rho$ (see Art. 6) will be as indicated in the figure. If this difference is divided by the scalar Δt (see Art. 4) we shall have a new vector $\Delta\rho/\Delta t$ having the same direction as $\Delta\rho$ but differing in magnitude. If now we let Δt approach zero, $\Delta\rho$ will also approach zero, but $\Delta\rho/\Delta t$ will in general approach a *vector* value differing from zero in magnitude and having a definite direction. This limiting vector is called the

vector velocity of the particle at the beginning of the interval Δt. It may also be called the derivative of the position vector ρ with respect to the time. Its direction is along a tangent to the curve (from the definition of a tangent). Its magnitude is the speed of the particle.

145. Vector Acceleration.—In any curvilinear motion the velocity vector is itself constantly changing, perhaps not in magnitude, but certainly in direction. It will itself have a vector

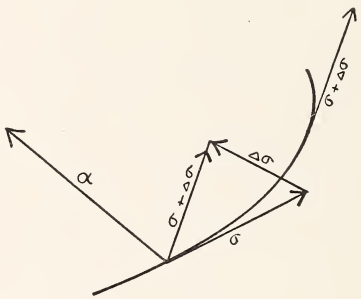


FIG. 211.

derivative formed in the same way. Thus let σ and $\sigma + \Delta\sigma$ (Fig. 211) be the velocity vectors at the beginning and at the end of a time interval Δt. Form the vector difference $\Delta\sigma$, divide it by Δt, and let Δt approach zero. The limit of the quotient $\Delta\sigma/\Delta t$ is the vector acceleration of the moving particle. It is evident that it must point somewhere toward the inner or concave side of the curve.

146. Composition and Resolution of Velocities and Accelerations.—If the curve we have considered lies in a plane, and we take coordinate axes in that plane with an origin at O, the components of ρ in the directions of the axes are the coordinates, x and y, of the moving particle (see Fig. 212). It is also clear

that the components of $\Delta\rho$ are Δx and Δy, those of $\Delta\rho/\Delta t$ are

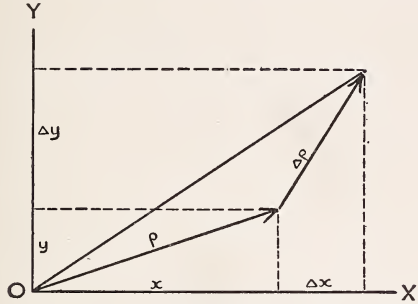


FIG. 212.

$\Delta x/\Delta t$ and $\Delta y/\Delta t$, and, finally, passing to the limit, the components of σ or $d\rho/dt$ are dx/dt and dy/dt. These are called v_x and v_y, respectively, or the x and y components of the velocity (see Fig. 213). If in Fig. 212 we replace ρ, x, y, $\Delta\rho$, Δx, and Δy, by σ, v_x, v_y, $\Delta\sigma$, Δv_x, and Δv_y, respectively, we can show in like manner that the components of α or $d\sigma/dt$ are

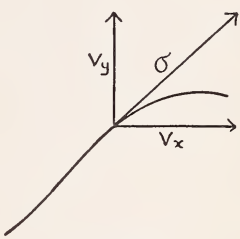


FIG. 213.

$$\alpha_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \text{ and } \alpha_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}.$$

The magnitude of σ, *i. e.*, the speed of the particle, is called v, and is equal to $\sqrt{v_x^2 + v_y^2}$. It is shown in the Calculus that $v = ds/dt$, where s is the distance measured along the curve. Similarly the magnitude of the acceleration α is $\sqrt{\alpha_x^2 + \alpha_y^2}$.

It is most important for the student to remember that in curvilinear motion *velocity and acceleration are both vectors*, and that neither is determined by giving its magnitude alone. It is no

more correct to identify a velocity of 10 feet per second east with a velocity of 10 feet per second north than it would be to identify the former with a velocity of 20 feet per second east.

Usually the most convenient way to specify the direction of a vector is to give the angle that it makes with a known direction, such as the x-axis. Thus the angle that α makes with the x-axis is $\tan^{-1}\frac{a_y}{a_x}$. In determining the quadrant of the angle, however, we must take account of the signs both in the numerator and in the denominator separately, not merely of the sign of the fraction. Thus if the components of a vector σ are $v_x = -\sqrt{3}$ and $v_y = +3$, its angle with OX is $120°$; but if $v_x = +\sqrt{3}$ and $v_y = -3$, the angle is $300°$. This matter will always be perfectly clear if we draw a figure.

Example 1.—A point moves so that $x = t^2$ and $y = \log t$. Find its velocity and acceleration when $t = 1$.

Solution.—Differentiating,

$$v_x = 2t = 2, \qquad a_x = 2,$$
$$v_y = \frac{1}{t} = 1, \qquad a_y = -\frac{1}{t^2} = -1,$$
$$v = \sqrt{4+1} = \sqrt{5}, \qquad a = \sqrt{5},$$

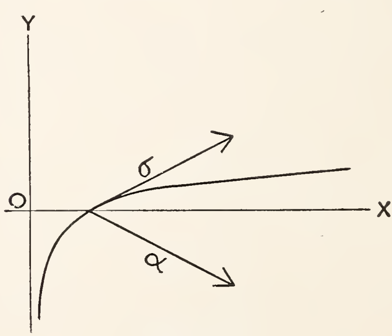


FIG. 214.

if we let a denote the magnitude of the acceleration. Here the velocity and acceleration are not equal although their magnitudes are the same. For their directions are different, σ pointing into the first quadrant, α into the fourth (see Fig. 214). If we denote by τ and ψ, respectively, the angles that the vectors σ and α make with OX, we shall have $\tau = 26°34'$, and $\psi = 333°26'$.

Example 2.—A particle P moves in the curve $y^2 = 4x$ so that v_x is always 2 ft/sec. Find v and α when P is 2 feet from the x-axis.

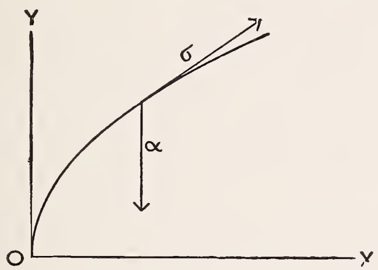


FIG. 215.

Solution.—

$$y^2 = 4x,$$
$$2y\,dy = 4dx.$$

Hence,

$$\frac{dy}{dt} = \frac{2}{y} \cdot \frac{dx}{dt}, \text{ or } v_y = \frac{2v_x}{y} = 2;$$
$$v = \sqrt{2^2 + 2^2} = 2\sqrt{2}\text{ f/s.} \quad \tau = 45°;$$
$$v_x = 2; \quad \therefore a_x = 0;$$
$$v_y = \frac{4}{y}; \quad \therefore a_y = -\frac{4}{y^2} \cdot \frac{dy}{dt} = -2.$$

Hence,

$$\alpha = 2 \text{ f/s}^2, \ \psi = 270°. \quad \text{(See Fig. 215.)}$$

Problems

518. A particle moves in a curve $y^2 = x^3$ so that $v_y = 4$ ft/sec. Find v and α when $y = 8$.

519. A point describes a curve $6y=x^2$, and has a speed of 10 ft/sec when $x=6$. Find the components of the velocity at that instant.

520. A particle moves in the path $y=3x^2$ so that the speed of its projection on OY is always 4 ft/sec. Find the direction of its motion 12 seconds after it passes the origin.

521. A particle describes the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$, so that $v_x=8$ f/s. Find v_y and a_y when $y=2$ feet. Find also the time of describing one-half of the ellipse.

522. Given $v_x=2y$, $v_y=2x$, show that the path is an equilateral hyperbola, and that $a_x=4x$ and $a_y=4y$.

523. If a particle moves so that $x=e^t+e^{-t}$ and $y=e^t-e^{-t}$, show that the velocity and the acceleration have each the same magnitude as the position vector at any time t. Show also that $\tau=90°-\psi$.

147. Velocity and Acceleration Independent of the Origin.—In defining vector velocity we considered a fixed point O as origin of the position vectors ρ and $\rho+\Delta\rho$. This was merely to preserve the analogy with the ordinary procedure of the Calculus. The vector $\Delta\rho$, however, is the vector from the position of the particle at the beginning of the time Δt to its position at the end; and this is certainly the same regardless of the origin. As the origin was not used again in forming σ and a, these vectors are entirely independent of the origin, and depend only on the motion itself. At any stage of an investigation, therefore, we are at liberty to assume an origin, and if desirable a set of coordinate axes, in any way that we please. If we desire the components of a vector in two given perpendicular directions, we may take coordinate axes in these two directions. We have seen that the velocity vector is always tangent to the curve. The acceleration vector, however, is not; and one of the most important problems is to determine its components along a tangent and a normal to the curve.

148. Tangential and Normal Acceleration.—To find a_t and a_n, the tangential and normal components of the acceleration at any point, take the origin at that point and the x-axis tangent to the path-curve (Fig. 216). Then *at this point*

$$v_y = \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = 0 \cdot v_x = 0,$$

and therefore $v = v_x$ *at this point.* In general

$$v^2 = v_x^2 + v_y^2.$$

Differentiating with respect to t,

$$2v\frac{dv}{dt} = 2v_x\frac{dv_x}{dt} + 2v_y\frac{dv_y}{dt}.$$

But, since $v = v_x$ and $v_y = 0$, we have

$$\frac{dv}{dt} = \frac{dv_x}{dt} = a_x = a_t = \frac{d^2s}{dt^2}.$$

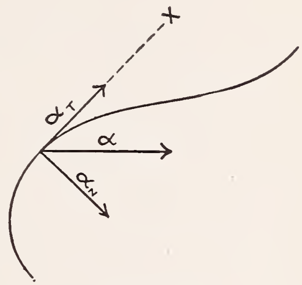


FIG. 216.

Thus the *tangential acceleration is the rate of change of the speed in the path.* It does not depend on the shape of the path. It is zero if the speed is constant.

To get the normal acceleration, we start with

$$v_y = \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

Differentiating,

$$a_n = a_y = \frac{dv_y}{dt} = \frac{dy}{dx} \cdot \frac{d^2x}{dt^2} + \frac{d^2y}{dx^2} \cdot \left(\frac{dx}{dt}\right)^2$$

$$= 0 \cdot a_x + \frac{1}{r} \cdot v^2,$$

since the expression for curvature, $1/r$, developed in the Calculus, reduces to $\frac{d^2y}{dx^2}$ when $\frac{dy}{dx} = 0$. The reason that we make this substitution is in order to get a result independent of our choice of axes.

We have then the very important expression for the normal acceleration,

$$a_n = \frac{v^2}{r},$$

where r is the radius of curvature at the point considered. Thus we see that the normal acceleration depends on the speed and the shape of the path, but not on the rate at which the speed is changing.

It will be seen that the expressions that we have just derived for a_t and a_n are not expressed in terms of coordinates. They are therefore valid regardless of the coordinates used. The student should try to visualize the acceleration vector, observing that it always points toward the inner side of the curve, that it points along the normal to the curve if the speed is constant, ahead of the normal if the speed is increasing, behind the normal if the speed is decreasing.

Example.—A point moves so that $x = 2t^2 + 3$, $y = t^2 + 2t$. Find the tangential and normal accelerations when $t = 2$.

Solution.—

$$x = 2t^2 + 3, \qquad y = t^2 + 2t,$$

$$v_x = 4t, \qquad v_y = 2t + 2,$$

$$a_x = 4, \qquad a_y = 2.$$

$$v^2 = v_x^2 + v_y^2 = 20t^2 + 8t + 4,$$

$$a_t = \frac{dv}{dt} = \left[\frac{20t+4}{v}\right]_{t=2} = 4.4. \quad \text{Ans.}$$

$$a = \sqrt{4^2 + 2^2} = 2\sqrt{5};$$

Also, $a = \sqrt{a_t^2 + a_n^2}$.

Hence, $a_n = \sqrt{a^2 - a_t^2} = \sqrt{20 - 19.36} = 0.8.$ Ans.

Problems

524. A particle moves so that $x = \cos 2t + \sin 2t$, $y = \cos 2t - \sin 2t$. Find the tangential and normal acceleration when $t = \pi/4$.

525. A point moves in the curve $y^2 = 4x$ so that $v_x = 2$ f/s. Find the components of the acceleration when the point is 2 feet above the x-axis.

149. Circular Motion.—If a particle moves in a circle, its position at any instant is determined by the angle, θ, from a fixed line through the center of the circle, to a line joining the center to the particle. The rate of change in $\theta, \frac{d\theta}{dt}$, is called the angular velocity and is commonly denoted by ω. If ω varies, its rate of change is $\frac{d\omega}{dt}$ or $\frac{d^2\theta}{dt^2}$. This is called the angular acceleration and will be denoted by β. The four variables t, θ, ω, and β, bear a perfect analogy to t, s, v, and a, respectively, in rectilinear motion. The formulas connecting the elements of the latter set may be converted into the formulas connecting the elements of the former, without any further proof, but always subject to the corresponding restrictions. Thus in rectilinear motion, $v^2 = v_0^2 + 2as$ *if the acceleration is constant.* So in angular motion, $\omega^2 = \omega_0^2 + 2\beta\theta$ (ω_0 being the initial angular velocity) *if the angular acceleration is constant.*

Problem 525a.—By the principle of correspondence just explained, write the following formulas for angular motion: (1) connecting time, angle, and angular velocity, when the last is constant; (2) time, angle, initial angular velocity, and angular acceleration, when the last is constant; (3) t, θ, ω_0, and ω, when β is constant; (4) ω, ω_0, β, and t, when β is constant; (5) t, ω_0, ω, and β, when β is variable.

150. Units.—In applying these formulas there is no restriction on the units employed except that they should be consistent. Thus if t is in hours and θ in revolutions, ω will be in revolutions per hour; or if ω is in radians per second, θ must be in radians and t in seconds. If the data are partly in one unit and partly in another, they must in general be made uniform before applying the formulas.

Example 1.—A body changes its angular velocity from 4 R. P. M. (revolutions per minute) to 40 R. P. M. in an interval

of 3 minutes. What is the angular acceleration (supposed uniform), and how many revolutions will the body make in the next two minutes?

Solution.—

$$\beta = \frac{\omega - \omega_0}{t} = \frac{40-4}{3} = 12 \text{ rev/min}^2.$$

For the next 2 minutes

$$\theta = \omega_0 t + \tfrac{1}{2}\beta t^2 = 40 \times 2 + 6 \times 4 = 104 \text{ revolutions.}$$

Problems

526. A fly-wheel starting from rest acquires a speed of 180 R. P. M. in making $7\frac{1}{2}$ complete revolutions. Find the time and the angular acceleration.

527. A motor is at rest and is brought to a velocity of 300 radians per minute in $\frac{1}{2}$ minute. Find the angular acceleration necessary and the number of revolutions made.

528. A "merry-go-round" making 100 R. P. M. is brought to rest in 2 minutes. Find the angular acceleration and the angle turned through before coming to rest.

529. A wheel is running at a uniform speed of 32 turns a second when a resistance begins to retard its motion uniformly at a rate of 8 radians per second per second; (a) how many turns will it make before stopping? (b) in what time is it brought to rest?

151. Linear and Angular Displacement, Velocity, and Acceleration.—If a particle moves in a circular path, the distance that it moves is determined if we know the angle turned through and the radius of the circle. If θ is the angle *in radians* (Fig. 217), r the radius, and s the length of the arc passed over, then

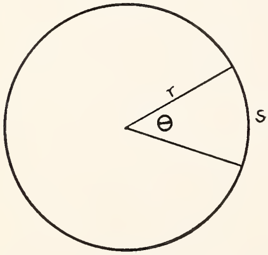


FIG. 217.

$$s = r\theta.$$

Since r is a constant, differentiating this twice with respect to t gives

$$v = r\omega \text{ and } a_t = r\beta.$$

In both of these, also, it must be remembered that the angular velocity and acceleration must be expressed *in terms of radians.* The student should contrast this restriction with the freedom of choice of units in the preceding article, and remember that *radians* must be used if the angle is to be multiplied by the *radius.* Also it should be remembered that $r\beta$ gives the tangential and not the total acceleration. The other component, the normal acceleration, being equal to v^2/r, may here be written $\omega^2 r$.

Example 1.—A particle moves on the circumference of a circle with the constant angular velocity ω. If the radius of the circle is r, find the magnitude and direction of the velocity and of the acceleration.

Solution.—From the above the magnitude of the velocity is $r\omega$. Its direction is of course tangent to the circle. We have just seen that the normal acceleration is equal to $\omega^2 r$. Since ω is constant, $\beta=0$ and the tangential acceleration is zero. Hence the total acceleration is $\omega^2 r$ and is directed along the normal, *i. e.*, toward the center of the circle.

Example 2.—Solve the preceding example by expressing x and y in terms of t, and finding the velocity and acceleration from their components parallel to the axes.

Solution.—Since the angular velocity is constant, $\theta=\omega t$, if we take $t=0$ when $\theta=0$. We then have

$$x=r\cos\theta=r\cos\omega t. \qquad y=r\sin\theta=r\sin\omega t.$$

$$v_x=-r\omega\sin\omega t. \qquad v_y=r\omega\cos\omega t.$$

$$a_x=-r\omega^2\cos\omega t. \qquad a_y=-r\omega^2\sin\omega t.$$

$$v=\sqrt{v_x^2+v_y^2}=r\omega.$$

$$a=\sqrt{a_x^2+a_y^2}=r\omega^2.$$

$$\tan\tau=\frac{r\omega\cos\omega t}{-r\omega\sin\omega t}=-\cot\omega t;\quad \tau=\omega t+90^\circ=\theta+90^\circ;$$

$$\tan\psi=\frac{-r\omega^2\sin\omega t}{-r\omega^2\cos\omega t}=\tan\omega t=\tan\theta.$$

But each component of a is opposite in sign to the corresponding coordinate. Consequently $\psi=\theta+180^\circ$; *i. e.*, the acceleration vector points toward the center of the circle. These results agree of course with the preceding solution.

Problems

530. For motion in a circle, $x=a\cos kt$, $y=a\sin kt$, find a_x, a_y, a_t, and a_n.

531. If a point moves on an ellipse $x=4\cos t$, $y=3\sin t$, show that $a_t/a_n=-\frac{7}{24}\sin 2t$.

532. A point moves in a circle of radius 10 feet with a uniform speed of 8 f/s. Find v_x and v_y when $\theta=30°$.

533. A point moves in a circle of radius 5 feet with a uniform speed of 6 f/s. Find the linear acceleration.

534. A 14-foot fly-wheel is making 80 R. P. M. Find the linear velocity and the linear acceleration of a point on the rim.

535. Taking the radius of the earth as 4000 miles, and $\pi=22/7$, find the linear velocity and linear acceleration of a point on the earth's equator.

536. A Ferris wheel of 15 feet radius, starting from rest, makes 1/10 of a revolution during the first second of its motion. Find the angular acceleration of the wheel (supposed constant). Find also the linear velocity and the total linear acceleration of a point on the rim at $2\frac{1}{2}$ seconds from rest.

537. The velocity of a particle moving in a circular path whose radius is 3 feet, changes from 7 revolutions to 4 revolutions per second in 5 seconds. If the acceleration is uniform:

(a) What is the angular acceleration?

(b) How many revolutions will the particle make before coming to rest?

(c) What was the normal acceleration at the beginning?

538. If a particle moves in a circle with uniform angular velocity about the center, show that its angular velocity about any fixed point in the circumference is also uniform, and is equal to one-half of that about the center.

152. Newton's Laws and the Differential Equation of Motion.— Up to this point in the study of curvilinear motion, we have been concerned only with *kinematics, i. e.,* the discussion of the motion itself without reference to the forces producing it. For the *dynamics* of curvilinear motion, *i. e.,* the study of the motion in relation to the forces producing it, we must generalize the state-

ment of Newton's Laws and the differential equation of motion. (See Arts. 115, 118.)

In curvilinear motion the First Law has obviously no application; and the Third Law presents no new difficulty. The Second Law must be restated, and this can best be done in terms of vectors. We have seen that velocity is a vector quantity. Momentum is also a vector quantity, being obtained by multiplying the velocity vector by the scalar factor of mass. The same will be true, of course, about change of momentum. We have also seen that force is a vector quantity. The Second Law may now be stated in the following form: *the two vector quantities, force and change of momentum, are proportional, i. e.,* one is a scalar constant times the other. The statement in terms of vectors includes without explicit mention the latter part of the law, which states that the change of momentum is in the same direction that the force acts. Of course we must make the same convention about constant force and unit time that we did in rectilinear motion. (Art. 118.) These restrictions are removed when we pass to the differential equation of motion. This equation, $\lambda F = m\alpha$, holds without change, if we merely understand that F and α are vectors. Of course m and λ are still scalars, the latter being simply a constant depending on the units used.

153. Components of Force and Acceleration.—Since the force vector and the acceleration vector have the same direction, which makes, let us say, an angle θ with the x-axis, the resolved part of F in the x direction will be $F \cos \theta$, and that of α will be $\alpha \cos \theta$. Denoting these components by X and α_x respectively, we shall get

$$\lambda X = \lambda F \cos \theta = m\alpha \cos \theta = m\alpha_x = m \cdot \frac{d^2x}{dt^2}.$$

This is just the same equation of motion that we should get if the force X acted alone. As the x-axis may be chosen at will, this gives us the very important result that the resolved part *in any*

direction of the total force acting on a body produces the same acceleration *in that direction* as if that resolved part were the only force acting.

This principle furnishes a practical method for solving a great many problems on motion in a plane. A set of rectangular axes is chosen, usually horizontal and vertical, and the differential equation of motion in the direction of each of these axes is integrated separately. Of course the initial velocity is a vector and must be resolved into its horizontal and vertical components to get the initial velocities in these two directions.

154. Free Motion and Constrained Motion.—There are two principal types of problem in curvilinear motion in a plane. In the first, that of *free motion,* the total force acting on the particle is known, and one part of the problem is to find the path which the particle describes. In the second, that of *constrained motion,* the particle is compelled to move along a certain path, which is known in advance, and one part of the problem is to find the force which must act on the particle to keep it in this path. Thus if a ball is thrown into the air (not vertically) we know the forces acting on it, namely gravity and air-resistance; but we do not know before solving the problem what the path is. This is an illustration of free motion. If, on the other hand, the ball is projected into a fixed tube, we know that its path must follow this tube; but we do not know before solving the problem what force the wall of the tube must exert on the ball in order to keep it in this path. This is an example of constrained motion. Other familiar examples of constrained motion are the sling-shot, the governor, a train moving on a curved track, etc.

In free motion we generally know a_x and a_y. Integrating these and determining the constants by means of the initial conditions, we get x and y in terms of t. The elimination of t between these two equations gives us a single equation in x and y. This is the equation of the path curve.

155. Motion of a Projectile.—As an illustration of free motion we shall now consider the motion of a particle projected obliquely upward from a given point O, with a given initial velocity. Let the xy-plane be the plane of motion (Fig. 218), OX and OY the horizontal and vertical axes; and let the particle be projected from O with an initial velocity of 1000 f/s directed along a line

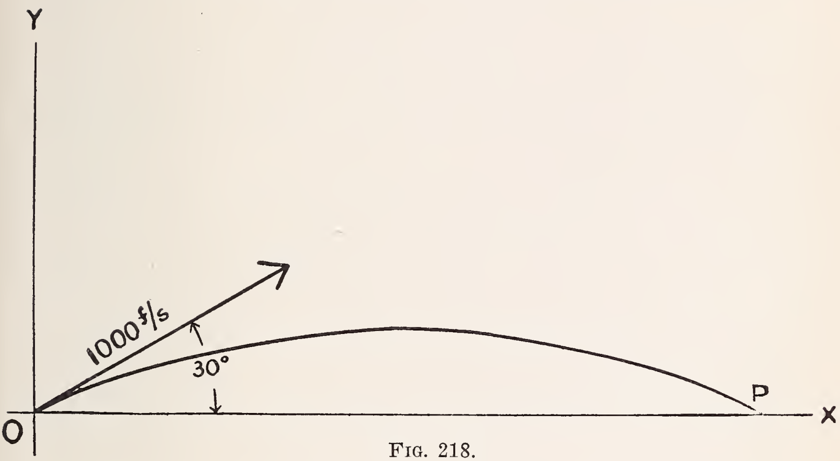


FIG. 218.

which is inclined at an angle of 30° to OX. If we neglect the resistance of the air, gravity is the only force acting on the body during the flight. The acceleration in the horizontal direction is therefore zero, and that in the vertical direction is $-g$. Hence,

$$a_x = \frac{d^2x}{dt^2} = 0, \text{ and } a_y = \frac{d^2y}{dt^2} = -g.$$

Integrating,

$$v_x = \frac{dx}{dt} = C_1, \quad v_y = \frac{dy}{dt} = -gt + C_2.$$

When $t=0$, the component velocities are 1000 cos 30° and 1000 ×

sin 30°. Substituting these values in the last pair of equations, we have

$$1000 \cos 30° = C_1, \quad 1000 \sin 30° = 0 + C_2,$$

and hence,

$$v_x = \frac{dx}{dt} = 500\sqrt{3}, \quad v_y = \frac{dy}{dt} = -gt + 500.$$

Integrating again,

$$x = 500\sqrt{3}t, \quad y = -\frac{gt^2}{2} + 500t. \qquad (1)$$

(The constants of integration are zero since $x = y = 0$ when $t = 0$.)

These equations expressing x and y in terms of t are the equations of motion of the projectile. They determine completely its position at any instant.

The student should notice that in these equations each element that determines the motion appears separately. The expression for x contains only the one term due to the horizontal component of the initial velocity. This is the same as for uniform motion, since there is no horizontal acceleration. The expression for y contains a similar term, $500t$, which would appear alone if gravity did not act. But it also contains the term $-\frac{1}{2}gt^2$, which would appear alone if the initial velocity were zero. Thus it is always easy to write the equations of motion for any projectile without going back to the velocity or acceleration, if we know the initial velocity.

If we wish the equation of the path, we can eliminate t, and get

$$y = \frac{x}{\sqrt{3}} - \frac{gx^2}{1{,}500{,}000},$$

which is the equation of a parabola. Thus the path of the projectile is parabolic.

To find the point P where the projectile will strike the x-axis in its descent (OP is called the *range* on the level), set $y = 0$ in (1) and solve, getting first

$$t = \frac{1000}{32} \text{ seconds,}$$

which is the time of flight (taking $g=32$). Substituting this value of t in the equation for x, we get

$$x=500\sqrt{3}\,\frac{1000}{32}=27{,}100 \text{ ft.},$$

which is the range.

To find the highest point of the trajectory, set $v_y=0$. This gives

$$t=\frac{500}{32}=15.6 \text{ secs.}$$

which is the time it takes the projectile to reach the highest point in its trajectory. Putting this value of t in the equation for y, we get

$$y=500\cdot\frac{500}{32}-\frac{32}{2}\left[\frac{500}{32}\right]^2=3900 \text{ ft.},$$

which is the height of the trajectory.

Problems

539. A projectile is fired with an initial velocity of 320 feet per second in a direction making an angle of 60° with the horizontal. Find the range.

540. Find the coordinates of the highest point reached by a body projected with a velocity of 256 f/s at an angle of 30° with the horizontal.

541. Find the value of the angle which the initial-velocity vector should make with the horizontal in order that the range should be a maximum. (*Hint:* Express the range in terms of this angle and the initial velocity v_0, and then differentiate, regarding v_0 as a constant).

542. A projectile fired from the top of a tower at an angle of elevation of 45° strikes the level ground 60 feet from the foot of the tower at the end of 5 seconds. Find the initial velocity and the height of the tower.

543. A body is projected with a velocity of 100 f/s at an angle of 75° with the horizontal. Find its range on a horizontal plane.

544. A shot is fired at an angle of 45° with the horizontal and an initial velocity of 192 f/s. Find the greatest height and the time to reach that height.

545. An airplane is moving horizontally with a velocity of 96 f/s, when a body is dropped from the machine. Find the speed of this body 4 seconds after being dropped, and the angle between its motion and that of the airplane at that instant.

546. What must be the initial velocity of a projectile to hit an object 3000 feet away and 408 feet high, if fired at an angle of 40° with the horizontal?

547. A man standing 15 feet from the foot of a pole 150 feet high, aims at the top of the pole. If the bullet just misses the top, where will it strike the ground if $v_0 = 1000$ f/s?

548. If a golf ball be driven from a tee horizontally with an initial speed of 300 f/s, where and when will it land on ground 16 feet below the tee (neglecting air resistance).

549. The velocity of sound is 1100 f/s. Find the range of a gun if the projectile and the sound of the discharge reach a point at the same instant.

550. The big gun used in the World War projected its shell to a height of about 24 miles. If the distance from Laon to Paris is 76 miles, what was the muzzle velocity of the shell and the angle of elevation? (The student must for the present neglect air resistance; but it should be remembered that the result is then only a very rough approximation.)

551. A piece of ice is detached on a roof whose inclination is 30°, at a point 8 feet from the eaves, which are 24 feet above the ground. At what distance from the vertical plane through the eaves will it reach the ground?

156. Centripetal and Centrifugal Force.—We have seen that any particle moving in a curve has an acceleration toward the inner side of the curve. That is, the total acceleration in curvilinear motion always has a normal component, which we have called the normal acceleration. From Newton's Second Law, or from the equation of motion, we know that there must be a force producing this, that is, the total force must have a normal component proportional to the magnitude of this acceleration and

to the mass of the moving particle. This normal component is called the *centripetal force.* If we are dealing with constrained motion, and this centripetal force is exerted by the constraint upon the moving particle, then there is an equal and opposite reaction exerted by the moving particle upon the constraint. This is called the *centrifugal force.* Thus if a stone is tied to a string and whirled around in a circle, the string exerts a force upon the stone, which continually deflects it from a straight-line path and compels it to move in a circle. This is the centripetal force. The reaction to this, namely, the force which the stone exerts upon the string, is the centrifugal force.

We know (Art. 148) that the normal acceleration is v^2/r, where r is the radius of curvature of the path at the point considered. If we take the equation of motion in the form $gF=Wa$ (Art. 125), we have that the centripetal force is

$$F_n = \frac{W}{g} a_n = \frac{W}{g} \cdot \frac{v^2}{r},$$

directed toward the inner side of the curve. The centrifugal force, which is the one more often mentioned, is of course equal and opposite.

If the path is circular, then r is simply the radius of the circle. In this case it will frequently be convenient to use $\omega^2 r$ instead of v^2/r for the normal acceleration.

Example.—An 8-pound weight is whirled in a circle on a smooth table at the end of a 4-foot cord which can just stand a tension of 9 pounds without breaking. Find the R. P. M. it is making at the instant that the cord breaks.

Solution.—

$$F_n = 9 = \frac{Wv^2}{gr} = \frac{8}{32} \cdot \frac{v^2}{4},$$

$$\therefore v = 12 \text{ f/s, and } \omega = 3 \text{ rad/sec},$$

or 28.7 R. P. M.

Problems

552. Find the centrifugal force of a weight of 64 pounds which is moving in a circle of 6-foot radius at a uniform speed so as to make one revolution in 3 seconds.

553. A weight of 2 pounds is connected by a string 2 feet long to a fixed point of a smooth table. If the string can sustain a tension of 8 pounds, what is the greatest number of revolutions per second that may be made without breaking the string?

554. A flat horizontal plate revolves about a vertical axis, making 1 revolution per second. If $\mu = 11/18$, what is the greatest distance from the axis at which a weight can lie on it and not be thrown off by the centrifugal force?

555. Assuming $g = 32$ f/s^2 and the radius of the earth 4000 miles, in what time could a body revolve freely around the earth close to its surface without being pulled down by gravity?

157. The Conical Pendulum.—A string attached to a fixed point and supporting a weight W which moves in a horizontal circle with uniform speed, constitutes the simplest case of the conical pendulum. In order that the weight may actually move in a circle, there must clearly be some relation between its speed and the angle that the string makes with the vertical. Let it be required to find this relation, and also the tension in the string.

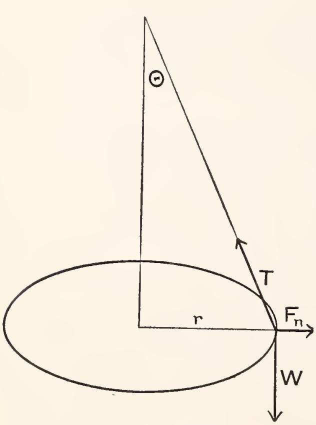


FIG. 219.

There are actually only two forces acting on the body, its weight W, and the tension in the string T. (Fig. 219.) The resultant of these must be just equal to the *centripetal* force necessary to produce the circular motion. Now the *centrifugal* force is equal and opposite to the centripetal, and hence equal and opposite to this resultant. Consequently it forms with this resultant a system of forces in equilibrium—or, what amounts to the same thing, the three forces, T, W, and the centrifugal force, are in equilibrium. Calling the last F_n,

and applying the conditions of equilibrium by resolving horizontally and vertically, we have

$$T \sin \theta = F_n,$$
$$T \cos \theta = W;$$

or, replacing F_n by $\omega^2 r W/g$,

$$\tan \theta = \frac{F_n}{W} = \frac{\omega^2 r}{g}.$$

Notice that the factor W cancels out and that $\tan \theta$ is merely the quotient of the two accelerations, the centrifugal, $\omega^2 r$, which is horizontal, and that of gravity, g, which is vertical. This is an easy way to remember this result, and it is equally easy to express $\tan \theta$ in terms of the linear speed v instead of ω, by using v^2/r instead of $\omega^2 r$. It is left as an exercise for the student to find the tension T.

In the solution of this problem we have an example of a very important method by which a problem in kinetics is replaced by a problem in statics, by the device of introducing a force equal and opposite to that necessary to produce the motion. This force will then be in equilibrium with the forces actually operating; and the problem is thus reduced to one in statics.

Example 1.—The string of a conical pendulum is 12 feet long and makes an angle of 60° with the vertical. Find the tension in the string and the linear velocity of the bob, if the latter weighs 5 pounds. (Fig. 220.)

Solution.—Resolving vertically,

$$T \cdot \tfrac{1}{2} = 5,$$
$$\therefore T = 10 \text{ lbs.}$$

Resolving horizontally,

$$T \cdot \tfrac{1}{2}\sqrt{3} = \frac{5}{32} \cdot \frac{v^2}{6\sqrt{3}},$$
$$\therefore v = 24 \text{ f/s.}$$

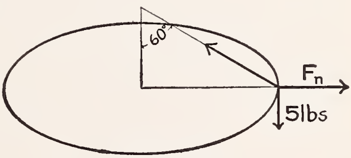


FIG. 220.

Example 2.—A conical pendulum makes $38\tfrac{2}{11}$ R. P. M. The length of the string is 30 inches and the weight of the bob is

4 pounds. Find the radius of the circle described, and also the tension in the string. (Take $\pi = 22/7$.)

Solution.—

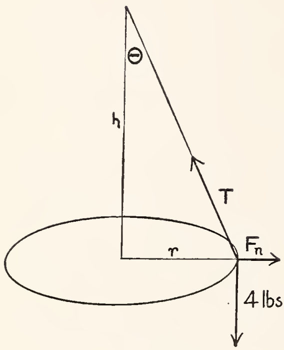


FIG. 221.

$$\omega = \frac{38\frac{2}{11} \cdot 2\pi}{60} = 4 \text{ rad/sec.}$$

Resolving horizontally,

$$T \sin \theta = \frac{4\omega^2 r}{g} = 2r.$$

Resolving vertically,

$$T \cos \theta = 4,$$

$$\therefore \tan \theta = \frac{r}{2}.$$

But from Fig. 221

$$\tan \theta = \frac{r}{h}.$$

Hence,
$$\frac{r}{h} = \frac{r}{2}$$

$h = 2$ ft. $= 24$ inches,
$r = 18$ inches,
$T = 4 \sec \theta = 4 \times \frac{5}{4} = 5$ lbs.

Problems

556. The cord of a conical pendulum is 3 feet long, and the bob, weighing 6 pounds, is making 40 R. P. M. Find the angle the cord makes with the vertical.

557. A 40-pound weight swings at the end of a 6-foot string which is tied to a hook in the ceiling. How far below the ceiling is the weight when making 50 R. P. M. and what is the pull on the string? (Take $\pi = 22/7$.)

558. A stone weighing 1 pound is whirled round by means of a string so as to describe a horizontal circle in a plane 2 feet below the point of suspension. Find the time of revolution and also the tension, the length of the string being 1 foot.

559. A plummet is suspended from the roof of a railway car. How much will it be deflected from the vertical when the train is running 45 miles per hour over a curve of 300 yards radius?

560. In the preceding problem what angle should the plane of the track make with the horizontal in order that the wheel pressure of the train should be normal to the track? If the distance between the rails is 4 feet 8 inches, how much should the outer rail be elevated to give this angle.

561. If d is the distance between the rails and h is the elevation of the outer rail, show that (approximately)

$$h = \frac{v^2 d}{gr},$$

if the pressure of the wheels is to be normal to the track. The quantities v, g, and r, have their usual meaning and should all be expressed in feet and seconds. The approximation consists in assuming the sine and tangent of a small angle equal to each other, the inclination of the track to the horizontal being so small in all ordinary cases that this is amply accurate for practical purposes.

562. Show that in the preceding problem d may be taken in any convenient units (different from v, g, and r), and that then h will be given in the same units as d.

563. A railroad curve has a radius of 1000 feet. Trains are to round the curve at 30 mi/hr. If the gauge is 4 feet 8 inches, how much must the outer rail be raised above the level of the inner in order that there shall be no lateral thrust on the rails?

564. A locomotive weighing 175 tons moves in an 800-foot curve with a speed of 40 mi/hr. Find the horizontal pressure on the rails if they are on the same level.

158. Surface of a Rotating Liquid.—The problem of finding the shape of the surface of a liquid subjected to rotation is one arising in the study of hydromechanics. If an open vessel containing a liquid rotates with uniform velocity about a vertical axis through its center, the liquid will soon acquire an angular velocity equal to that of the vessel, as a result of friction. The free surface will assume a concave shape, symmetrical with respect to the axis of rotation; and we shall now show that this surface is a paraboloid of revolution.

Consider a vertical section of the vessel, OY being the axis of rotation, and x and y the coordinates of a particle of weight W

in the free surface of the liquid. (Fig. 222.) Acting on this particle we have the weight W, which is vertical, and the centrifugal force, which is horizontal. By Art. 74, the surface of the liquid must stand at right angles to the resultant R of these forces, so that R is perpendicular to the tangent as shown.

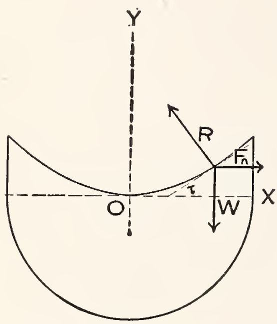


FIG. 222.

For a given angular velocity, we have, then,

$$R \sin \tau = \frac{W\omega^2 x}{g},$$
$$R \cos \tau = W,$$
$$\therefore \tan \tau = \frac{\omega^2 x}{g}.$$

Now by Calculus

$$\tan \tau = \frac{dy}{dx},$$
$$\therefore \frac{dy}{dx} = \frac{\omega^2 x}{g},$$

and, integrating,

$$y = \frac{\omega^2 x^2}{2g},$$

the constant of integration being zero if we take the origin at the lowest point of the curve. Since this is the equation of a parabola, the surface of the liquid is a paraboloid of revolution.

Example.—If a hemispherical bowl of radius 28 inches is filled with liquid and made to rotate about its vertical radius at 45 R. P. M. how much will overflow?

Solution.—The paraboloid of revolution in this case must pass through the edge of the bowl; and the required volume is that of the portion of this paraboloid cut off by the plane of this edge. This volume is (by Calculus) equal to half the volume of the circumscribed cylinder. The radius of this is the radius of the

bowl, 7/3 (in feet if we use $g=32$). Its altitude is the value of y in the equation of the parabola when $x=7/3$,

$$y=\frac{\omega^2x^2}{2g}=\frac{(\frac{3}{2}\pi)^2(\frac{7}{3})^2}{64}=\frac{49\pi^2}{256}.$$

Hence,

$$V=\tfrac{1}{2}\pi x^2y=\tfrac{1}{2}\pi\frac{49}{9}\cdot\frac{49\pi^2}{256}=16.2 \text{ cu. ft.}$$

Problems

565. A hollow paraboloid of revolution, altitude 2 feet and radius of base 8 inches, with axis vertical and vertex down, is half filled with liquid. What must its angular velocity about its axis be in order that the liquid may just rise to the rim?

566. Find the pressure on the top of a closed cylinder of radius r, just full of liquid of density w, and rotating uniformly about its vertical axis with a velocity ω.

567. A hemispherical bowl of radius r, containing a given volume of water V, is set rotating about a vertical radius. At what angular velocity does the water begin to overflow?

159. The Simple Pendulum.—A heavy particle suspended from a fixed point by a weightless, inextensible cord, and caused to move in an arc of a vertical circle under the action of gravity, is called a simple pendulum. The particle is called the pendulum *bob*. We may regard the forces acting on the bob as its weight W, the tension of the cord T, and the centrifugal force F_n due to the motion. The directions in which these forces act are indicated in the figure.

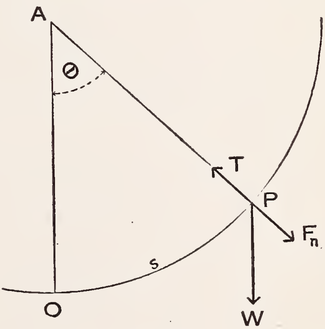


FIG. 223.

Denote by A the fixed point of suspension, by l the length of the cord, by P the position of the particle at any time, by θ the angle between the vertical AO and the cord AP, and by s the length of the arc OP.

Suppose the pendulum bob is drawn aside from its vertical position through an angle α and then released. Since no motion takes place perpendicular to the path, we have at P

$$T = W \cos\theta + F_n.$$

The only force which affects the motion is the component along the tangent to the path at P of the resultant of all the forces that act on the particle. This component is $W \sin\theta$. Since this force tends to decrease the distance s, the equation of motion of the pendulum bob is

$$W\frac{d^2s}{dt^2} = -gW\sin\theta,$$

or, since $s = l\theta$,

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta.$$

This differential equation may be integrated by multiplying both members by $2\frac{d\theta}{dt}dt$:

$$2\frac{d^2\theta}{dt^2}\cdot\frac{d\theta}{dt}\cdot dt = -2\frac{g}{l}\sin\theta d\theta.$$

This gives on integration

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{l}\cos\theta + C_1.$$

Since $\frac{d\theta}{dt} = 0$ when $\theta = \alpha$, $C_1 = -\frac{2g}{l}\cos\alpha$,

and therefore

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{l}(\cos\theta - \cos\alpha).$$

It follows, then, that

$$t = \sqrt{\frac{l}{2g}}\int\frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}},$$

an integral which can not be expressed in terms of the functions used in elementary mathematics.

However, when the angle of vibration of the pendulum is small, a close approximation to the motion is obtained by replacing $\sin\theta$ by θ in the equation of motion. This then becomes

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta,$$

which can be integrated completely. Since g and l are positive constants, this is the differential equation of simple harmonic motion. (Art. 133.) The time t of a single oscillation is then

$$t = \pi\sqrt{l/g}.$$

Thus the period is independent of the length of the arc through which the pendulum swings.

160. The Seconds Pendulum.—A pendulum which makes one oscillation in one second is called a seconds pendulum. The length of such a pendulum is

$$l = \frac{g}{\pi^2}.$$

If the value of g is taken as 32.17, the length of the seconds pendulum is

$$l = \frac{32.17}{\pi^2} \text{ feet} = 39.12 \text{ inches}.$$

If the length l of a pendulum which beats seconds at a given place is known, the value of the acceleration of gravity at this place is

$$g = \pi^2 l.$$

This is the method regularly used for determining the exact values of g at different points on the earth's surface.

161. The Cycloidal Pendulum.—A particle of weight W suspended from a fixed point A by a weightless, inextensible, and flexible cord of length l, and caused to move in a cycloid under

the action of gravity, is called a cycloidal pendulum. The forces acting on the particle are its weight W, the centrifugal force F_n due to its motion, and the tension T in the cord. The directions in which these forces act are indicated in the figure. A cycloidal pendulum may be constructed by causing the cord of the pendulum to wind on another cycloid.

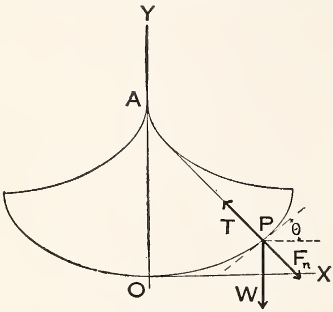


FIG. 224.

Denote by P the position of the particle at any instant, by θ the inclination to the horizontal of the tangent to the path at P, and by s the length of the arc OP.

Since no motion takes place perpendicular to the path, we have

$$T = W \cos\theta + F_n.$$

The only force which affects the motion is $W \sin\theta$, a force which tends to decrease s. Therefore the equation of motion is

$$W\frac{d^2s}{dt^2} = -gW\sin\theta,$$

or

$$\frac{d^2s}{dt^2} = -g\sin\theta.$$

The equations of the cycloid may be written in the form

$$x = a(\phi + \sin\phi),$$
$$y = a(1 - \cos\phi),$$

and it can easily be shown by Calculus that $a = l/4$ and $\phi = 2\theta$. Therefore,

$$ds = \sqrt{dx^2 + dy^2} = \frac{l}{4}\sqrt{4(1+\cos 2\theta)^2 + 4\sin^2 2\theta}\,d\theta = l\cos\theta\, d\theta,$$

and

$$s = l\int_0^\theta \cos\theta\, d\theta = l\sin\theta.$$

The equation of motion may now be written in the form

$$\frac{d^2s}{dt^2}=-\frac{g}{l}s,$$

which is the differential equation of simple harmonic motion. (Art. 133.) Therefore the time t of a single oscillation of a cycloidal pendulum is

$$t=\pi\sqrt{l/g}.$$

Thus the period is independent of the arc through which the pendulum swings. It should be noted that this statement is exact for the cycloidal pendulum, as there was no approximation made in integrating its equation of motion. The corresponding statement made in the case of the simple pendulum was an approximation, holding good only for small amplitudes.

Problems

568. Find the length of a seconds pendulum where $g=32.20$.

569. Find the value of g at a place where the length of the seconds pendulum is 39.06 inches.

570. How many oscillations per minute will a 40-inch pendulum make at a place where $g=32.16$?

571. If 39.11 inches is the length of the seconds pendulum at a certain place, how long must a pendulum be to make 10 oscillations a minute at that place?

572. A pendulum oscillates 50 times per minute at one place, and 50.2 times at another. Compare the values of g at the two places.

162. Motion Along a Smooth Curve in a Vertical Plane.—Consider a particle of weight W starting from A with an initial velocity v_0 and sliding down a smooth curve in a vertical plane, as shown in the figure. Denote by P the position of the particle at any instant, and by τ the angle that the tangent to the curve

makes with the horizontal. The tangential force is then $W \sin \tau$, and the equation of motion is therefore

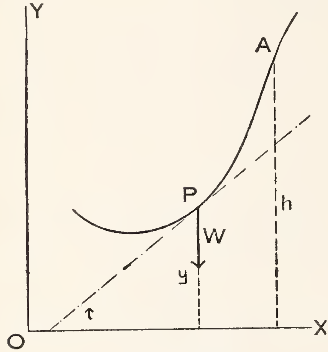


FIG. 225.

$$W \sin \tau = \frac{W}{g} \cdot \frac{d^2s}{dt^2}.$$

Since

$$\sin \tau = -\frac{dy}{ds}, \text{ and } \frac{d^2s}{dt^2} = v\frac{dv}{ds},$$

we have

$$\frac{W}{g} v \frac{dv}{ds} = -W\frac{dy}{ds}.$$

Integrating,

$$\frac{v^2}{2} = -gy + C.$$

Since the particle started from A with a velocity v_0, $v=v_0$ when $y=h$, h being the original height, so that $C=\frac{v_0^2}{2}+gh$. Therefore

$$v^2 = v_0^2 + 2g(h-y).$$

The distance $h-y$ being the vertical projection of the path, this equation expresses the fact that the velocity acquired by a particle sliding down a smooth curve in a vertical plane is the same as if it had fallen freely through the corresponding vertical distance.

Example 1.—A particle slides down a smooth vertical circular arc, starting 60° from the lowest point. Find the velocity at the lowest point if the radius of the arc is 6 inches.

Solution.—

$$h = BC = \tfrac{1}{4} \text{ ft};$$
$$v^2 = 2gh = 16;$$
$$\therefore v = 4 \text{ f/s}.$$

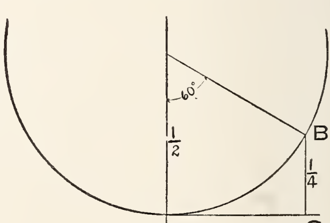


FIG. 226.

Example 2.—A weight is whirled in a vertical circle at the end of a cord 3 feet long, having just sufficient velocity to travel in the circle. What is the velocity at the highest and lowest points of the circle?

Solution.—At the highest point the centrifugal force must be just enough to support the weight, or the centrifugal acceleration must be equal to the acceleration of gravity, or $v^2/r=g$. Hence, $v^2/3=32$, or $v=4\sqrt{6}$ f/s, at the highest point. At the lowest point the velocity will exceed this by the same amount as if it had fallen freely a distance equal to the diameter of the circle. Hence at the lowest point

$$v^2=v_0^2+2gh=96+2\times32\times6=480.$$
$$\therefore v=4\sqrt{30}\text{ f/s.}$$

Problems

573. Find the velocity of the weight in the preceding example when on a level with the center of the circle.

574. A pail of water is caused to swing in a vertical circle at the end of a cord 4 feet long. If the water is to remain in the pail find the necessary velocity at the highest and lowest points.

575. If the pail and the water in the preceding problem together weigh 6 pounds, find the tension in the cord when the pail is at a point 60° from the lowest point of the circle.

576. A body is projected from the lowest point of a vertical circular track of radius 6 feet. If the body is prevented from leaving the track, what velocity at the lowest point will carry it around?

577. By the method of this article derive an expression for the linear velocity of the bob of the simple pendulum at any point of its swing, and from this derive the formula for its angular velocity given in Art. 159.

163. Work.—If any number of forces, F_1, F_2 . . . . , variable in magnitude and direction, act on a particle, which moves under their influence along some curved path, and if the (variable) angles which the lines of action of these forces make with the path are θ_1, θ_2 . . . . , then the work done by the several forces will be respectively $\int F_1 \cos\theta_1\, ds$, $\int F_2 \cos\theta_2\, ds$ . . . . , the integration

extending over the whole path. If the resultant of all these forces is F, making an angle θ with the path, then its resolved part, $F\cos\theta$, along the path will be equal at each point to the sum of the resolved parts, $F_1\cos\theta_1$ . . . . , of the components. Hence,

$$\int F\cos\theta\,ds = \int F_1\cos\theta_1\,ds + \int F_2\cos\theta_2\,ds + \ .\ .\ .\ .$$

or

The quantity of work done by the resultant of any number of forces is equal to the algebraic sum of the quantities of work done by the several forces during the same displacement.

164. Kinetic Energy.—The total work done upon a body moving in any curved path may be found in terms of its initial and final velocities. For by Art. 153, $F\cos\theta = m\dfrac{dv}{dt}$. Hence,

$$\int_{S_0}^{S_1} F\cos\theta\,ds = \int_{S_0}^{S_1} m\cdot\frac{dv}{dt}\,ds = \int_{S_0}^{S_1} m\cdot\frac{ds}{dt}\cdot\frac{dv}{ds}\,ds =$$

$$\int_{v_0}^{v_1} mv\,dv = \frac{mv_1^2}{2} - \frac{mv_0^2}{2}.$$

Thus we see that in the most general case of plane curvilinear motion the work done upon a particle in changing its motion is equal to the increase in its kinetic energy, or the work done by the particle is equal to the decrease in its kinetic energy.

This principle combined with the preceding, which enables us to consider separately the work done by each force that acts, is particularly useful in considering constrained motion. Here the reaction of the constraint, which is the force that keeps the particle in its path, does no work; for it acts at every point in a direction perpendicular to that of the motion. The change in kinetic energy, therefore, is equal to the work done by or against the remaining force or forces.

Example 1.—The bob of a 5-foot pendulum weighs 2 pounds. If it starts from rest at its lowest point and is acted on by a constant horizontal force of 6 pounds, what velocity will it have when the cord is horizontal?

Solution.—In this motion work is done *by* the horizontal force, and work is done *against* the force of gravity. The only other force acting is the tension in the cord, and this does no work. Each of the first two forces acts through a distance of 5 feet, the radius of the circle. Hence the work-energy equation becomes

$$6 \times 5 - 2 \times 5 = \tfrac{1}{2} \cdot \frac{2}{g} \cdot v^2.$$

Hence,

$$v = \sqrt{20g} = 25.4 \text{ f/s.}$$

Example 2.—The bob of a simple pendulum of length 10 feet is drawn aside until it is 4 feet above the horizontal line through the lowest point of its path, and is then released. If the bob weighs 5 pounds, find the tension T in the cord when the bob is 1 foot above the horizontal line.

Solution.—The forces acting on the bob are as indicated in the figure. Resolving forces along the cord, we get

$$T = 5 \cos \theta + \frac{5}{g} \cdot \frac{v^2}{10}.$$

From the work-energy equation,

$$5(4-1) = \frac{5}{2g}(v^2 - 0^2),$$

and

$$v^2 = 6g.$$

Therefore

$$T = 5\left(\tfrac{9}{10}\right) + \frac{5}{g} \cdot \frac{6g}{10} = 7.5 \text{ lbs.}$$

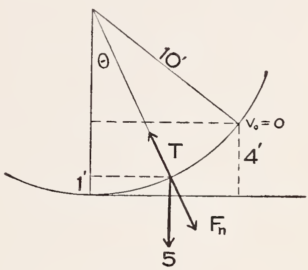


FIG. 227.

Problems

578. What will be the angle with the horizontal made by the cord of the pendulum in Example 1 when the bob reaches its maximum elevation? (*Hint:* The kinetic energy will then be zero, and the work done can easily be expressed in terms of the angle.)

579. Show that if a pendulum bob, starting from rest at its lowest point, is acted on by any constant horizontal force, it will rise to a point where the angle the cord makes with the vertical

is twice as great as that when the bob is in equilibrium under the action of gravity and the horizontal force.

580. If two equal bodies are projected with the same velocity at the two angles which give the same horizontal range, show that the sum of the kinetic energies at their highest points is independent of the angles of projection.

581. A particle slides down the outside of a smooth vertical circle, starting from rest at the highest point of the circle. Find where it will meet the horizontal plane through the lowest point of the circle.

582. A particle starting from a point (a, b) moves so that v_x and v_y vary as the corresponding coordinates; find the equation of the path and the accelerations along the axes.

583. In uniform circular motion show that the angular velocity about any point in the circumference is also uniform, and equal to one-half of what it is about the center.

584. A particle describes the hyperbola $x^2/9-y^2/16=1$, find (a) a_x if $v_y=8$, (b) a_y if $v_x=12$.

585. A particle describes a parabola with such a varying velocity that its projection on a line perpendicular to the axis of the parabola is a constant k. Find the velocity and acceleration parallel to the axis of the parabola.

586. Show that the velocity vector of a point describing a cycloid passes through the highest point of the generating circle.

587. A chandelier weighing 80 pounds is suspended from the ceiling of a hall by means of a chain 12 feet long whose weight is neglected. By how much is the tension of the chain increased if it be set swinging so that the velocity at the lowest point is 6 f/s?

588. A stone slides 12 feet without friction down a roof of inclination 30°. If the lower edge of the roof be 50 feet above the ground, (a) when, (b) where, and (c) with what velocity will the stone strike the ground?

589. A stone is dropped from a balloon, which, at a height of 625 feet, is carried along by a horizontal air current at the rate of 15 m/h. Where, when, and with what velocity will it reach the ground?

590. A particle P moves in a curve $y^2=x^3$ so that $v_y=4$ f/s. Find a_t and a_n when $y=8$.

591. Show that a projectile whose elevation is 60° rises three times as high as when its elevation is 30°, the magnitude of the initial velocity being the same in each case.

592. A wheel 6 feet in diameter is making 50 R. P. M. when thrown out of gear. If it comes to rest in 4 minutes, find (a) the angular retardation, (b) the linear velocity of a point on the rim at the beginning of the retarded motion, (c) the same after 2 minutes.

593. A projectile fired from the top of a tower at an angle of elevation of 45° strikes the ground 60 feet from the foot of the tower at the end of 4 seconds. Find the height of the tower.

594. A stone thrown with a velocity of 64 f/s is to hit an object on top of a wall 19 feet high and 48 feet distant. Determine the direction of the initial velocity.

595. Find expressions for v_r and v_θ where r and θ are the polar coordinates of a point.

596. A spring is compressed from 20 to 17 inches by a weight of 471 pounds. How much work must be done to compress it another inch?

597. A body whose weight is 64.4 pounds falls freely from rest from a height of 5 feet upon a 200-pound helical spring. Find the compression in the spring. (A 200-pound spring is one that is compressed 1 inch by a force of 200 pounds.)

598. Prove that the statement, "a rifle bullet does not rise more than one inch in a range of 100 yards," implies that the initial velocity must be greater than 2078 f/s.

CHAPTER XI

DYNAMICS OF A RIGID BODY

165. Rigid Body.—As stated in Chapter III we shall regard a rigid body as a collection of particles so connected that the distance between any two of them is constant. If the body is continuous we may regard it as made up of an infinite number of elements of mass dm and we may perform any summation extending over all the particles by integration.

166. Plane Motion.—We shall confine our attention to motion in which all the particles of the body move along curves parallel to one fixed plane. This is called *plane motion.* We may think of the body as made up of laminæ or thin flat slices parallel to this plane. These laminæ are all rigidly connected, and hence, if we know the motion of one of them, we know the motion of the entire body. A revolving fly-wheel furnishes an illustration of plane motion. Each particle moves in a fixed plane perpendicular to the axis of the wheel. We may also consider the wheel to be made of thin plates joined rigidly together, which rotate in planes perpendicular to the axis of the wheel. As a second illustration consider a car travelling along a straight track. Each particle moves in a fixed vertical plane. A particle in one of the car wheels rotates about a point on the axle which is itself moving forward, and the motion takes places in one of the fixed planes described.

167. External and Internal Forces.—In certain cases, as that of a body acted on by gravity only, the forces acting tend to produce the same acceleration in each of the particles, *i. e.,* each particle would have the same acceleration even if it were not rigidly connected with the others. In most instances, however, change of motion in a rigid body is produced by the application

of forces to only some of the particles. Such forces are called *external* forces. To illustrate, consider the motion of a railroad coach on a straight track. The pull of the drawbar, which is applied at one point, moves all the particles of the car without changing their relative positions. We know that it takes a force to accelerate any one of the particles and this force must originate from the drawbar, and be transmitted to the particle. Thus there must be forces acting between the particles of the body. These are called *internal* forces. They are the forces which resist the deformation of the body and keep it rigid.

In dealing with internal forces we shall assume that the forces acting between two particles of the body have the same line of action and are equal and opposite. Thus in any summation of all forces acting on all particles of the body, all internal forces will cancel and leave only the external forces. In Chapter IV we have the truss, an example of a rigid body at rest, acted on by a system of external and internal forces. The loads and supports constitute a system of external forces applied to certain of the particles; the stresses, induced by the loading in the various members of the truss, constitute a system of internal forces; further, the force exerted by any member on a pin is exactly equal and opposite to the force that the pin exerts on this member and has the same line of action.

168. Translation.—If a body moves so that every line in it remains parallel to itself the motion is called translation. The particles will move in parallel curves. A common illustration is that of a block sliding down a plane with all its particles moving in straight lines, but in any motion of a compass box on a table the needle remains parallel to itself and its motion is translation. The particles of the needle describe curved paths. In translation all particles of a body have at any instant the same velocity and the same acceleration.

169. Motion of the Center of Gravity in Translation.—Suppose a body of mass M to have the motion of translation. Let dm be the mass of an elemental particle of the body, a its acceleration, and dF the resultant of all the forces acting on dm. Then $dF = adm$. As stated, it follows from the definition of translation that all particles of the body have the same acceleration at any instant. Hence the several forces dF are parallel forces, acting parallel to the direction of the acceleration a, and the resultant of the system of external forces acting on the body acts in the same direction, and is

$$\int dF = \int adm = a\int dm = aM,$$

since the internal forces cancel in the summation. To find the point of application (x_0, y_0, z_0) of the resultant we assume a set of coordinate axes (as in Art. 56) and by taking moments about them obtain

$$x_0 \cdot a \cdot M = \int x \cdot a \cdot dm, \; y_0 \cdot a \cdot M = \int y \cdot a . dm, \; z_0 \cdot a \cdot M = \int z \cdot a \cdot dm.$$

Since a is constant for the body at any instant we may cancel it out. We then recognize these equations as the equations (4) which determine the coordinates of the center of gravity.

Therefore, if a body has the motion of pure translation, the system of external forces acting on it is equivalent to a single resultant force acting at the center of gravity in the direction of the acceleration. Also, the center of gravity moves as if the whole mass were concentrated there and acted on by the resultant.

170. Thus it follows that the solution of problems involving the translation of a rigid body is reduced to the consideration of a particle at the center of gravity of the body, whose mass equals that of the body, and which is acted upon by a force that is the resultant of all the forces acting on the body. The following problems can be solved by principles already studied in earlier chapters. In particular we can use the principles of work and energy as applied in Chapter IX to the particle.

Example 1.—What is the effective H. P. of a locomotive which gives a 500-ton train a speed of 15 m/h up a grade of 1 in 320, frictional resistance being 12 pounds per ton?

Solution.—

$$v = 15 \times 22/15 = 22 \text{ f/s.}$$

$500 \times 2240 \times \frac{1}{320} \times 22 =$ work in ft.-lbs. in 1 sec. against gravity.

$12 \times 500 \times 22 =$ work in ft.-lbs. in 1 sec. against friction.

$$\therefore \text{H. P.} = (22 \times 12 \times 500 + 500 \times 2240 \times \tfrac{1}{320} \times 22)/550 = 380.$$

Example 2.—A 600-ton train moves down a grade of 1 in 50 with a uniform velocity after power has been shut off, and upon reaching the level it runs half a mile before coming to rest. If the resistance remains constant find the velocity of the train on the grade.

Solution.—Let $v =$ velocity down the grade in miles per hour,

$F =$ the resistance.

Then K. E. in ft.-lbs. at the bottom of the grade is

$$1/2 \times 600 \times 2240/32 \times (22v/15)^2.$$

Since there is just enough resistance to neutralize the accelerating force of gravity,

$$F = 600 \times 2240 \times 1/50.$$

Substituting in the work-energy relation,

$$F.s = \tfrac{1}{2}m(v^2 - v_0^2),$$

we have

$$600 \times 2240 \times 1/50 \times 2640 = 1/2 \times 600 \times 2240/32 \times (22v/15)^2;$$
$$\therefore v = 39 \cdot 6 \text{ m/h.}$$

Problems

599. A freight car weighing 40 tons and moving 12 m/h is brought to rest on a level track in a space of $\frac{1}{3}$ of a mile. Find the mean frictional resistance to the motion in pounds.

600. A coasting party of five people, whose average weight is 150 pounds, reaches the foot of a hill at a speed of 30 m/h. The sled weighs 50 pounds and $\mu = 1/10$; how far will they travel along the the level?

601. An engine is capable of producing 150,000 ft.-lbs. of energy per second; find the greatest uniform velocity at which it can draw a train weighing 100 tons along a level track, the frictional resistance being 1/90 of the weight.

602. A 500-ton train is moving up a grade of 1 in 95 against frictional resistance of 15 pounds per ton. What H. P. must the engine develop at the instant when the speed is 15 m/h if the acceleration of the train at that instant is $\frac{1}{5}$ f/s^2?

603. A freight train, weighing 200 tons and travelling 20 m/h, runs into an engine weighing 50 tons and standing on the track. Find the velocity with which the broken engine will be forced along the track if $e = \frac{1}{5}$.

604. An elastic cord, which stretches 1 inch under a pull of 5 pounds, is being used to raise a weight of 80 pounds through a height of 6 feet. If the work used in stretching the string is lost, what is the efficiency of this lifting device?

605. A 25-ton freight car, moving with a velocity of 4 m/h, strikes a bumping-post. Assuming that the post absorbs none of the shock, find the amount of compression of the 50,000-pound spring of the draft rigging.

606. A steam engine has a cylinder 2 feet long with diameter of 10 inches. Steam with boiler pressure of 150 pounds per square inch is admitted to the cylinder during the first half-stroke and the valve is then closed. If the steam pressure varies according to the law $p \cdot v = \text{const.}$, find the work done during the forward stroke. If the flywheel is making 200 R. P. M., what is the H. P. of the engine?

171. Rotation.—When a body turns about an axis fixed in the body and fixed in space we have simple rotation. Each particle moves along the arc of a circle whose plane is perpendicular to the axis of rotation and all particles have at any instant the same angular velocity and the same angular acceleration.

172. Equation of Angular Motion.—Let P be an element of mass dm of a rigid body which has the motion of simple rotation. Let OZ, Fig. 228, be the axis of rotation, r be the distance of

dm from the axis of rotation, β the angular acceleration of the body, and dF the sum of the tangential components of the forces acting on dm. The linear acceleration of P is then $r\beta$, and the equation of motion of P is

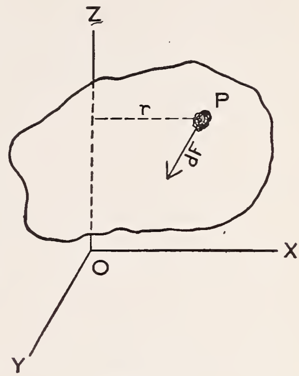


FIG. 228.

$$dF = r\beta dm.$$

Taking moments about OZ, we have

$$rdF = r^2\beta dm,$$

and, integrating to cover the entire mass,

$$\int r\,dF = \beta \cdot \int r^2 dm = I\beta,$$

where I is the moment of inertia of the body about the axis of rotation, and $\int rdF$ is the sum of the moments of the external forces acting on the body about the axis of rotation—it being remembered that the moments of the internal forces cancel in the summation. Denoting this moment by M_Z, we have

$$M_Z = I\beta.$$

That is: *The total moment of the external forces about any fixed axis of rotation is equal to the moment of inertia about that axis multiplied by the angular acceleration of the body.*

If we write the equation for the rotation of a body about a fixed axis, as $I = M_Z/\beta$, we see that I is the measure of the moment of the force necessary to produce a unit angular acceleration in overcoming the inertia of the body. Hence the origin of the name "moment of inertia" given to I. It is analogous to the mass, M, in the equation for rectilinear motion, $F/a = M$, where M is the measure of the force necessary to produce a unit linear acceleration in overcoming the inertia of the body.

173. *Illustrative Examples*

Example 1.—A cylinder, weight 200 pounds and radius 15 inches, is free to turn about its geometric axis. A fine cord is wound around it and a pull of 5 pounds is applied to the cord. Find the angular acceleration of the cylinder.

Solution.—$M_Z = 5 \times 5/4$ lb.-ft. = sum of the moments of the external forces.

$$I = 200/32 \cdot k^2 = 200/32 \cdot \tfrac{1}{2} \cdot (5/4)^2 \text{ lb.-ft.}^2$$

Substituting in the above formula,

$$5 \cdot 5/4 = 200/32 \cdot \tfrac{1}{2} \cdot 5/4 \cdot 5/4\ \beta,$$
$$\therefore \beta = 32/25 \text{ rad/sec.}^2$$

Example 2.—A solid cylindrical drum, weighing 175 pounds and 3 feet in diameter, is making 180 R. P. M. What tangential braking force will bring it to rest in 16 seconds?

Solution.—$F \cdot 3/2$ = moment of external forces about axis of rotation.

$$I = 175/32 \cdot \tfrac{1}{2} \cdot (3/2)^2;$$
$$\beta \cdot 16 = \omega = 180/60 \cdot 2\pi \text{ rad/sec} = \text{angular velocity.}$$
$$\therefore \beta = 3\pi/8 \text{ rad/sec.}$$
$$\tfrac{3}{2} F = \tfrac{175}{32} \cdot \tfrac{1}{2} \left(\tfrac{3}{2}\right)^2 \tfrac{3\pi}{8}, \quad \therefore F = 4.8 \text{ lbs.}$$

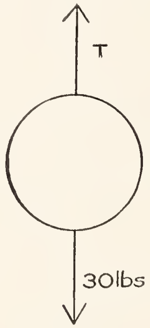


FIG. 229.

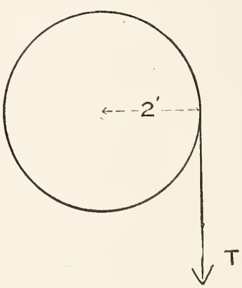


FIG. 230.

Example 3.—A metal drum, 4 feet in diameter and weighing 288 pounds, has wound on it a fine cord, to the end of which a weight of 30 pounds is attached. The drum revolves freely about

its axis, which is horizontal; the radius of gyration about this axis is $k=1.6$ feet. Find the tension in the cord.

Solution.—The equation of motion of the weight is (Fig. 229)

$$30-T=30/32\cdot a, \quad \therefore T=30(1-a/32);$$

also $$a=2\beta.$$

$$2T=\text{moment of external forces},$$

$$I=288/32\cdot(1.6)^2.$$

Substituting in $M_Z=I\beta$ (Fig. 230),

$$2T=288/32\cdot(1.6)^2\beta=288/32\cdot(1.6)^2\cdot a/2.$$

Solving simultaneously with the first equation,

$$T=25.8 \text{ lbs.}$$

Problems

607. A grindstone, 3 feet in diameter and weighing 200 pounds, is making 45 R. P. M. when a tangential braking force of 3 pounds is applied at the rim. How long will it take for the stone to come to rest?

608. A solid disk, weighing 980 pounds and 4 feet in diameter, is making 80 R. P. M. What tangential braking force applied at the rim will bring it to rest in 2 minutes 34 seconds?

609. A cylinder (with axis horizontal), 1 foot in diameter and weighing 100 pounds, has a fine cord wound on it, to the free end of which is attached a weight of 20 pounds. The cylinder is free to revolve about its axis. Find the linear and angular accelerations and the tension of the cord.

610. A solid homogeneous sphere of radius 1 foot and weight 2000 pounds is mounted on a horizontal weightless axle of diameter 4 inches, around which a small cord is wound carrying a weight of 50 pounds. Neglecting friction, find the angular acceleration of the sphere, the linear acceleration of the weight, the tension of the cord, and the velocity of the weight at the end of 5 seconds from rest.

611. A grindstone, 4 feet in diameter and weighing 225 pounds, is making 150 R. P. M. A tangential friction of 6 pounds is applied at the rim. How many complete revolutions will the stone make before coming to rest?

174. The Compound Pendulum.—A body which rotates about a fixed horizontal axis under the action of its weight is called a compound pendulum.

Pass a plane through G, the center of gravity of the body, perpendicular to the axis of rotation, and let the section be represented by Fig. 231. The axis of rotation is a line through O perpendicular to this plane. Let W = the weight of the body, k_0 = the radius of gyration about O, L = the distance OG, θ = the angle OG makes with the vertical. Since we may replace the gravitational forces by their resultant, W, acting through G, the moment of the external forces about the axis of rotation is $WL \sin \theta$. Then by the equation of angular motion,

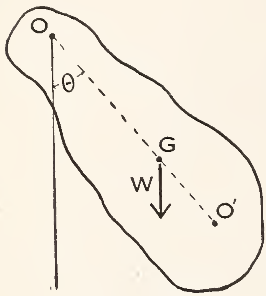


FIG. 231.

$$WL \sin \theta = I\beta = -\frac{W}{g} \cdot k_0^2 \cdot \frac{d^2\theta}{dt^2},$$

whence

$$\frac{d^2\theta}{dt^2} = \frac{-gL \sin \theta}{k_0^2}.$$

Placing the constant $L/k_0^2 = 1/l$, we get

$$\frac{d^2\theta}{dt^2} = \frac{-g \sin \theta}{l},$$

which is the differential equation that was obtained for the motion of the simple pendulum in Art. 159. Hence a compound pendulum will oscillate about an axis in exactly the same manner as a simple pendulum whose length is equal to the squared radius of gyration of the body about the axis of rotation divided by the distance of the center of gravity from the same axis.

175. Centers of Suspension and Oscillation.—The point O is called the center of suspension. The point O', obtained by

measuring the length k_0^2/L from O along OG, is called the center of oscillation.

Let k_G^2 = the radius of gyration of the body about an axis through the center of gravity parallel to the axis of rotation, then

$$k_0^2 = k_G^2 + L^2,$$

and

$$l = (k_G^2 + L^2)/L,$$

therefore,

$$L(l-L) = k_G^2.$$

Since L and $l-L$ enter the expression in exactly the same way, it follows that if O' were taken as the point of suspension, O would be the center of oscillation. Therefore, in a compound pendulum the centers of suspension and oscillation are interchangeable without altering the period.

Example.—Find the time of vibration for a small oscillation of a plank, 4 feet by 1 foot by 1 inch, about an axis perpendicular to the 4 foot by 1 foot face through the middle point of the 1 foot edge.

Solution.—

$k_0^2 = k_G^2 + 2^2 = 65/12$ ft.^{2}

$l = 65/24$ ft. = length of equivalent simple pendulum.

$t = \pi\sqrt{l/g} = \pi\sqrt{65/24 \cdot 1/32} = 0.91$ secs.

The center of oscillation is $17/24$ ft. from the center of gravity; therefore, if the axis were moved to a point $17/24$ ft. from the center of gravity, the time of vibration would not be changed.

Problems

612. A solid right circular cylinder, radius a, oscillates about a horizontal axis which is parallel to the geometric axis of the cylinder and at a distance of $3a/2$ from the latter. Find the time of a small vibration.

613. A thin uniform rod, 4 feet long and 1 inch in diameter, oscillates about a horizontal axis, 1 foot from one end and parallel to a diameter of one base. Find the length of the equivalent simple pendulum.

614. A cylindrical bar of 4 inches radius and 2 feet long is suspended from an axis which coincides with a diameter of one of its ends. If it is slightly disturbed how many oscillations will it make in a minute?

615. What must be the ratio of the radius of the base to the height of a right circular cone, in order that the center of oscillation may be in the base when the center of suspension is at the vertex?

616. A cast-iron sphere whose radius is 6 inches vibrates as a pendulum about a tangential line as an axis. Find the period of vibration and the length of a simple pendulum having the same period. Locate the center of oscillation.

617. A board 4 feet by 1 foot by 1 inch vibrates about an axis perpendicular to the 4 foot by 1 foot face through a point 18 inches from the center. Find the time of vibration for small oscillations and the length of the equivalent simple pendulum.

176. The Torsion Balance.—A torsion balance or pendulum consists of a rigid body suspended by a thin rod, the axis of which passes through the center of gravity of the body. The rod is rigidly joined to the body at the point of support and held rigidly at the other end also.

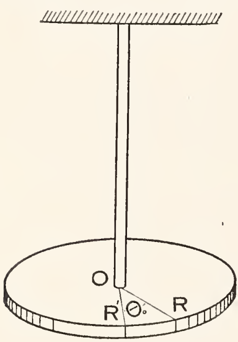


FIG. 232.

Suppose we have a torsion pendulum consisting of a suspended disk as shown in Fig. 232. Let OR be a radius marked on the upper surface of the disk at the position of rest. If the disk is turned through an angle θ_0 (OR will then take the position OR') the rod is twisted, and when the disk is released the rod exerts a twisting moment on the disk tending to return it to the initial position of rest. By the equation of angular motion, $M_Z = I\beta$. But in Art. 109 it was shown that the twisting moment exerted by the rod is proportional to the angle of twist. Hence we may write

$$I\beta = I\, d^2\theta/dt^2 = -k\theta,$$

which is recognized as the equation of simple harmonic motion when we substitute θ for s and k/I for k^2 in the equation of Art. 133, and proceeding as before we find

$$\omega = -\sqrt{k/I \cdot (\theta_0{}^2 - \theta^2)},$$

$$\arcsin(\theta/\theta_0) = \pi/2 - \sqrt{k/I}\, t,$$

and hence the time of one swing from the extreme left to the extreme right is

$$t = \pi\sqrt{I/k}.$$

If we know the twisting moment, M_1, exerted by the rod when twisted through any given angle θ_1, we have $M_1 = k\theta_1$, and hence

$$t = \pi\sqrt{I\theta_1/M_1}.$$

This equation gives the time of vibration independently of the initial displacement.

Example 1.—A uniform metal disk, 9 inches in diameter and weighing 6 pounds, is suspended from its center by a wire so that its plane is horizontal, and then twisted. What is the time of a complete oscillation if a twisting moment of 2 lb.-ft. causes a deflection of 12°?

Solution.—

$$M_1 = 2 \text{ lb.-ft.}$$
$$\theta_1 = 12 \times \pi/180 = \pi/15 \text{ rad.}$$
$$I = 6/32.2 \times \tfrac{1}{2} \times (\tfrac{3}{8})^2 \text{ lb.-ft.}^2$$
$$t = 2\pi\sqrt{I\theta_1/M_1} = \text{time of a complete vibration.}$$
$$\therefore t = 2\pi\sqrt{\tfrac{6}{32.2} \times \tfrac{1}{2} \times \tfrac{3}{8} \times \tfrac{3}{8} \times \tfrac{\pi}{15} \times \tfrac{1}{2}} = 0.233 \text{ secs.}$$

Example 2.—The moment of inertia of a torsion balance is 6300, where units are pounds and inches, and its time of vibration is 20 seconds. A disk L is fitted to the balance so that its center of gravity lies in the axis of suspension. The time of vibration is then found to be 30 seconds. Find the moment of inertia of L.

Solution.—Let I = the moment of inertia of the disk. Substituting in the formula,

$$t = \pi\sqrt{I/k};$$

before the disk is attached

$$20 = \pi\sqrt{6300/k};$$

after the disk is attached

$$30 = \pi\sqrt{(6300 + I)/k}.$$

Dividing one by the other,

$$\tfrac{2}{3} = \sqrt{6300/(6300 + I)}.$$

Solving,

$$I = 7875 \text{ lb.-in.}^2.$$

Problems

618. A fly-wheel of 3 tons is fastened to one end of a shaft, the other end of which is fixed, and the torsional rigidity of which is such that it twists 0.4° per ton-foot of twisting moment applied to the flywheel. If the radius of gyration of the flywheel and shaft combined is 3 feet, find the number of torsional vibrations per minute the wheel would make if slightly twisted and then released.

619. The moment of inertia of a torsion balance fitted with a platform is 7500 in lb.-ft units, and the time of vibration is 25 seconds. A body L placed on the platform with its center of gravity in the line of suspension, and the time of vibration is observed to be 30 seconds. Find the moment of inertia of L.

177. Kinetic Energy of Rotation and the Work-Energy Relation.—In a rigid body, let dm be the mass of an element which has linear velocity v and is at a distance r from the fixed axis of rotation. The kinetic energy of this element is $\frac{1}{2} \cdot dm \cdot v^2$, and that of the entire body is the sum $\frac{1}{2}\int v^2 \cdot dm$. Since $v = r\omega$, where ω is the angular velocity of the body, we have

$$\tfrac{1}{2}\int v^2 \cdot dm = \tfrac{1}{2}\int r^2\omega^2 \cdot dm = \tfrac{1}{2} \cdot I \cdot \omega^2,$$

that is, *The kinetic energy of a body rotating about a fixed axis is equal to the moment of inertia of the body about this axis multiplied by one-half the square of the angular velocity of the body.*

Let dF be the sum of the tangential components of all forces acting on the element of mass dm. By the equation of angular motion

$$\int r \cdot dF = M_Z = I \cdot \beta = I \cdot d^2\theta/dt^2,$$

which we may write

$$(\int r \cdot dF) \cdot d\theta = \tfrac{1}{2} I (2 \cdot d^2\theta/dt^2 \cdot d\theta/dt \cdot dt).$$

But $\omega = d\theta/dt$ and $d\omega = d^2\theta/dt^2 \cdot dt$. Substituting these values, we have

$$(\int r dF) d\theta = \tfrac{1}{2} I \cdot 2\omega \cdot d\omega;$$

integrating,

$$\int (\int r dF) d\theta = \tfrac{1}{2} I \omega^2 + C.$$

From Fig. 233 it is seen that the work done by dF as the body rotates through an angle $d\theta$ is $dF \cdot ds$, but $ds = r \cdot d\theta$, and the total work done by all the forces acting on the body is found by the summation $\int (\int r dF)\, d\theta$. If ω_0 denote the initial angular velocity, then when $\omega = \omega_0$, $\int (\int r dF)\, d\theta = 0$ (no work having been done) and therefore $C = -\tfrac{1}{2} I \omega_0^2$. Thus

$$\int (\int r dF) d\theta = \tfrac{1}{2} I (\omega^2 - \omega_0^2),$$

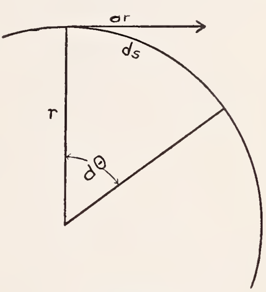


FIG. 233.

which equation states that: *The work done by the external forces* (since the internal forces cancel in the summation) *on a body rotating about a fixed axis is equal to the change in the kinetic energy of the body.* From this equation it also appears that if the body has an initial angular velocity ω, the work it can do before coming to rest is $\tfrac{1}{2} I \omega^2$, and hence $\tfrac{1}{2} I \omega^2$ is the kinetic energy of the body rotating about a fixed axis with angular velocity ω.

Example 1.—A cylinder, of weight 200 pounds and radius 15 inches, rotates about its geometric axis, making 120 R. P. M. A tangential force of 5 pounds is applied to its lateral surface. How many turns will the cylinder make before coming to rest?

Solution.—The work done against the tangential force in one revolution is $2\pi \cdot 5/4 \cdot 5$ ft.-lbs. For n revolutions the work done is $n \cdot 25/2 \cdot \pi$ ft.-lbs.

$$120/60 \cdot 2\pi = \text{the angular velocity in rads. per sec.}$$

Setting the total work done = the change in K. E.,

$$n \cdot 25/2 \cdot \pi = \tfrac{1}{2} \cdot 200/32 \cdot \tfrac{1}{2} \cdot (5/4)^2 \cdot (4\pi)^2;$$
$$\therefore n = 9\tfrac{23}{28} \text{ revolutions.}$$

Problems

620. A solid wheel, 2.6 feet in diameter and weighing 140 pounds, is making 50 R. P. M. Neglecting friction, find the amount of work in ft.-lbs. which must be done to increase its angular velocity to 70 R. P. M.

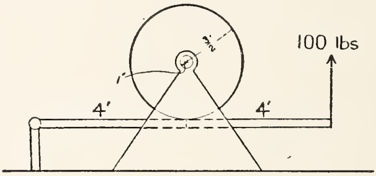


FIG. 234.

621. A solid cast-iron drum, of radius 3/2 feet and thickness 1 foot, is making 200 R. P. M. when a brake is applied and it is brought to rest. The brake is applied from below by a 100-pound force acting at the end of an 8-foot hinged beam (Fig. 234) which the drum touches 4 feet from the hinge. The coefficient of friction between the drum and the brake is $\frac{1}{4}$. The radius of the axle is 1 inch and the coefficient of axle friction is 1/20. How many revolutions will the drum make before coming to rest?

622. A fly-wheel, weighing 15 tons and of diameter 20 feet, is making 60 R. P. M. Find the energy stored up. If the axle of

the wheel is 14 inches in diameter and the coefficient of axle friction is $\frac{4}{5}$, when the wheel is disconnected how many revolutions will it make before coming to rest?

623. Find the kinetic energy of rotation of the earth, considering it a uniform sphere of density 5.6 and of diameter 8000 miles.

624. To control an engine against its own variations it is necessary to call upon the fly-wheel for 60,000 ft-lbs. of energy, and at the same time a change of velocity from 160 R. P. M. to 140 R. P. M. is allowable. If the wheel is 10 feet in diameter, compute its weight in tons.

178. Power Transmitted by Shafts.—In a shaft we have a cylinder rotating about its geometric axis and transmitting energy. Let b be the length of the crank arm and F the mean force acting on it through one revolution. Then the work done in one revolution is $2\pi b \cdot F = 2\pi \cdot T$, where T is the mean twisting moment. (See Art. 109.) To find the H. P. transmitted, we must express the work done in ft.-lbs. per minute and divide by 33,000.

Example 1.—A solid steel shaft, diameter 6 inches, is used to transmit power. It is to make 160 R. P. M. and the allowable stress is 5000 pounds per square inch. What H. P. may be transmitted?

Solution.—

$$T = q\pi r^3/2 = 5000 \cdot 22/7 \cdot 3^3/2 = 212,000 \text{ lb.-in.}$$
$$\text{H. P.} = 2\pi Tn/33000 = 2 \cdot 22/7 \cdot 70714/12 \cdot 160/33000 = 539.$$

Example 2.—The resistance of a twin-screw vessel at 18 knots is 44,000 pounds. At 95 R. P. M. what will be the twisting moment on each shaft? What is the H. P.?

Solution.—Let T = the mean twisting moment on one shaft in ton-ft. Then

$2 \cdot 2\pi T \cdot 95$ = the work done on the shafts in one minute in ft.-tons.
$18/60 \cdot 44000/2240 \cdot 6080$ = work done against resistance in ft.-tons in one minute.

H. P. $33000/2240$ = work done in one minute in ft.-tons.

Equating these quantities,

$$2 \cdot 2\pi T \cdot 95 = 18/60 \cdot 44000/2240 \cdot 6080 = \text{H. P. } 33000/2240;$$
$$\therefore T = 30 \text{ ton-ft., H. P.} = 2432.$$

Problems

625. What H. P. can a steel shaft 6 inches in diameter transmit at 100 R. P. M.?

626. Find the diameter of a solid shaft to transmit 9000 H. P. at 140 R. P. M., stress allowed being 10,000 pounds per square inch, and maximum twisting moment 3/2 the mean.

627. Find the size of a hollow steel shaft to replace the shaft of the preceding problem, the internal diameter being $\frac{9}{16}$ of the external. What is the saving in weight for 60 feet of shafting?

628. The pitch of a screw propeller is 14 feet, and the twisting moment on the shaft is 120 ton-in.; if the mean diameter of the thrust-bearing rings is 15 inches and the coefficient of friction is .05, find the thrust and the efficiency of the thrust bearing.

179. Translation and Rotation: Plane Motion.—In the general case of plane motion the center of gravity of the body moves (translates) in a plane and the particles of the body rotate about an axis through it perpendicular to the plane. Let the coordinates of G be $(\bar{x}, \bar{y})$, the coordinates of any particle P, of mass dm, be (x, y) and (r, θ) with respect to a set of axes through G parallel to the first set. (Fig. 235.) Then

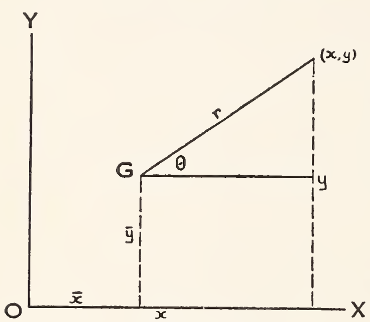


FIG. 235.

$$x = \bar{x} + r\cos\theta,$$
$$y = \bar{y} + r\sin\theta.$$

Differentiating with respect to t,

$$v_x = d\bar{x}/dt - r\sin\theta\, d\theta/dt,$$
$$v_y = d\bar{y}/dt + r\cos\theta\, d\theta/dt.$$

Differentiating again,

$$\begin{aligned} a_x &= d^2\bar{x}/dt^2 - r\sin\theta\, d^2\theta/dt^2 - r\cos\theta\cdot(d\theta/dt)^2,\\ a_y &= d^2\bar{y}/dt^2 + r\cos\theta\, d^2\theta/dt^2 - r\sin\theta\cdot(d\theta/dt)^2. \end{aligned} \tag{6}$$

The resultant of the forces acting on dm is determined by the components

$$dX = a_x dm, \quad dY = a_y dm;$$

and the resultant of all the external forces acting on the body—since the internal forces cancel in the summation—by

$$X = \int a_x dm, \quad Y = \int a_y dm.$$

Substituting from equation (6),

$$X = \int d^2\bar{x}/dt^2 \cdot dm - \int d^2\theta/dt^2 \cdot r \cdot \sin\theta \cdot dm - \int (d\theta/dt)^2 \cdot r \cdot \cos\theta \cdot dm;$$

$$Y = \int d^2\bar{y}/dt^2 \cdot dm + \int d^2\theta/dt^2 \cdot r \cdot \cos\theta \cdot dm - \int (d\theta/dt)^2 \cdot r \cdot \sin\theta \cdot dm.$$

But $\int r \cdot \sin\theta \cdot dm = \int y \cdot dm = 0$, $\int r \cos\theta . dm = \int x \cdot dm = 0$, since the origin is at G. (See Art. 56.) Therefore, since all the derivatives are constant for the integration,

$$\begin{aligned} X &= \int d^2\bar{x}/dt^2 \cdot dm = M \cdot d^2\bar{x}/dt^2, \\ Y &= \int d^2\bar{y}/dt^2 \cdot dm = M \cdot d^2\bar{y}/dt^2, \end{aligned} \qquad (7)$$

showing that the center of gravity of a body in plane motion moves as if the entire mass of the body were concentrated at that point and all the external forces were applied to it parallel to their original directions.

Taking moments about G of the forces acting on dm, we have

$$dM_Z = -r \cdot \sin\theta \cdot a_x \cdot dm + r \cdot \cos\theta \cdot a_y \cdot dm,$$

and, integrating, we find the sum of the moments of all the forces acting on the body:

$$\begin{aligned} M_Z = \int -r \cdot \sin\theta \cdot d^2\bar{x}/dt^2 \cdot dm + \int \sin^2\theta \cdot d^2\theta/dt^2 \cdot r^2 \cdot dm \\ + \int \sin\theta \cdot \cos\theta \cdot (d\theta/dt)^2\, r^2 \cdot dm + \int r \cdot \cos\theta \cdot d^2\bar{y}/dt^2 \cdot dm \\ + \int \cos^2\theta \cdot d^2\theta/dt^2 \cdot r^2 \cdot dm - \int \sin\theta \cos\theta\, (d\theta/dt)^2 \cdot r^2 \cdot dm. \end{aligned}$$

The first and fourth integrals are zero for the reason given above, the third and sixth cancel, and the equation reduces to

$$M_Z = d^2\theta/dt^2 \int r^2 \cdot dm = I_G \beta, \tag{8}$$

where I_G is the moment of inertia of the body about the axis of rotation through G.

From equations (7) and (8) we see that when a system of coplanar forces acts on a rigid body (in Art. 46 such a system was shown to be equivalent always to a single force and a single couple), the motion of G is governed by the magnitude and direction of the resultant force and is independent of its point of application, while the rotation of the body about the axis through the center of gravity is governed by the moment of the resultant couple and is independent of the motion of the center of gravity and the magnitude and direction of the resultant force.

The kinetic energy of a body in plane motion is the sum of the kinetic energy of translation and the kinetic energy of rotation. Thus if $\bar{v}$ denote the linear velocity of G we have

$$E = \tfrac{1}{2} \cdot M \cdot \bar{v}^2 + \tfrac{1}{2} \cdot I_G \cdot \omega^2.$$

The work done by the body is found by taking the sum of the following equations (see Art. 164):

$$\int_{x_0}^{x} X\, dx = M/2 \cdot (\bar{v}_x^2 - \bar{v}_{x_0}^2);$$

$$\int_{y_0}^{y} Y\, dy = M/2 \cdot (\bar{v}_y^2 - \bar{v}_{y_0}^2);$$

$$\int_{\theta_0}^{\theta} M_Z\, d\theta = I/2 \cdot (\omega^2 - \omega_0^2).$$

Thus, $\quad \text{Work} = M/2 \,.\, (\bar{v}^2 - \bar{v}_0^2) + \tfrac{1}{2} \cdot I_G \cdot (\omega^2 - \omega_0^2)$, or

The work done by a body in plane motion is equal to the change in the kinetic energy of translation plus the change in the kinetic energy of rotation about the axis through the center of gravity.

180. *Examples*

Example 1.—A right circular cylinder, weight 10 pounds and radius 3 inches, rests on end on a smooth table. A fine cord is wound around the cylinder and a force of 1 pound applied to the end of the cord. Determine the motion of the cylinder.

Solution.—Applying the two equal and opposite forces F' and F'' (Fig. 236) at G, we see that F' is the resultant force which will cause the cylinder to translate. Thus,

$$a = 32/10 = 3.2 \text{ f/s}^2.$$

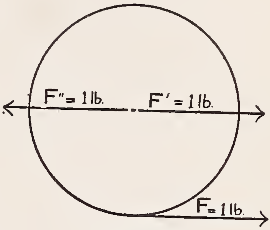


FIG. 236.

The couple F' F'' has a lever arm of 3 inches, and we find the angular acceleration from the equation $M = I_G\beta$.

$$\beta = (1 \times 3/12)/I_G = 3/12 \cdot 64/10 \cdot 12/3 \cdot 12/3 = 25.6 \text{ rad/sec.}^2$$

The center of gravity of the cylinder will move with an acceleration of 3.2 f/s² and the cylinder will rotate about its geometric axis with an acceleration of 25.6 rad/sec².

Example 2.—A cylinder, 2 feet in diameter and weighing 20 pounds, rolls down a rough plane 48 feet long and inclined 30° to the horizontal. If it starts from rest what is the angular velocity at the bottom?

Solution.—

$$20 \cdot 48 \cdot \tfrac{1}{2} = \text{work in ft.-lbs. done by the cylinder;}$$
$$\omega = v/r = v/1 = \text{angular velocity in rad. per sec.;}$$
$$\tfrac{1}{2} \cdot 20/32 \cdot v^2 = \text{gain in kinetic energy of translation;}$$
$$\tfrac{1}{2} \cdot 20/32 \cdot \tfrac{1}{2}\omega^2 = \text{gain in kinetic energy of rotation.}$$

Therefore

$$20 \cdot 48 \cdot \tfrac{1}{2} = \tfrac{1}{2} \cdot 20/32 \cdot v^2 + \tfrac{1}{2} \cdot 20/32 \cdot r^2/2 \cdot \omega^2$$
$$= \tfrac{3}{4} \cdot 20/32 \cdot v^2;$$
$$\therefore v = 32 \text{ f/s}, \quad \omega = v/r = 32 \text{ rad/sec.}$$

Problems

629. A cylinder rolls down a 30° incline. Find its velocity and the distance it has rolled at the end of 9 seconds. What is its velocity after it has rolled 18 feet from rest?

630. A solid 30-pound sphere rolls down a 30° incline in 7 seconds. Find the length of the incline.

631. A solid sphere, a disk, and a hoop start rolling from rest at the same time at the top of an incline. It what order will they reach the bottom?

632. A hoop rolls in a vertical plane. Show that the energy of rotation is one-half of the total kinetic energy.

633. The kinetic energy acquired by a sphere in sliding (without rolling) from rest down a smooth plane, is to that acquired by an equal sphere rolling (without sliding) down a rough plane by the same inclination and length, as 7 to 5.

Review Problems

634. Two solid cylinders, weights 400 pounds and 800 pounds, radii 1 foot and 2 feet respectively, are mounted on an axle coinciding with their geometric axes. Weightless cables are wound around these cylinders—the cable on the smaller being attached to a weight of 250 pounds, that on the larger to a weight of 300 pounds. If the system is initially at rest and friction is neglected, find: (a) the velocity of the 300-pound weight when it has moved 10 feet; (b) the angular velocity of the cylinders at that time; (c) the angular acceleration of the cylinders and the linear accelerations of the weights; (d) the time of descent.

635. A car weighs 40,000 pounds, the eight wheels weigh 4000 pounds, and the speed is 30 m/h. Find the kinetic energy stored up.

636. Show that "there is more energy stored in a ton of car-wheel than in a ton of car-body." (R. R. Gazette.) How much when the speed of the train is 40 m/h?

637. A bullet weighing 1 ounce is fired horizontally into a box of sand (inelastic) weighing 20 pounds, and remains imbedded in the sand. The box is suspended by a string attached to a fixed point 4 feet from the center of the box of sand. The impact of the bullet causes the box to swing aside through an angle of 42°.

Find the velocity of impact of the bullet, the kinetic energy lost in impact, the greatest tension in the string.

638. A weight W rests on a smooth table and is connected with a second weight H by a cord passing over a smooth pulley A. (Fig. 237.) When the motion starts, H is 7/2 feet above the floor. Find the velocity when H strikes the floor.

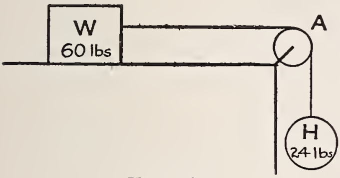


FIG. 237.

639. A solid cylinder, 18 inches in diameter and weighing 300 pounds, which revolves freely on its axis without friction, has a fine cord wound on it and a weight of 20 pounds attached to the end of the cord. The support under the 20-pound weight is removed and the weight descends under the action of gravity. What is the kinetic energy stored in the cylinder when the cord has caused the cylinder to make 3 complete revolutions?

640. A twin-screw steamship of 22,000 H. P. runs 3300 miles in 6 days. Find the resistance to the motion. At 95 R. P. M. what will be the mean twisting moment on each shaft?

641. A weight of 20 pounds rests on a rough table ($\mu = \frac{1}{6}$). A is a smooth pulley over which passes a string connecting W with the weight $P = 10$ pounds. The string is horizontal from W to A. Find the tension of the string. (Fig. 238.)

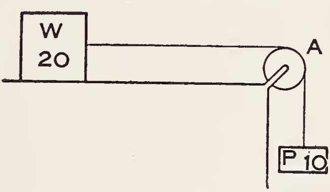


FIG. 238.

642. A 200-pound weight is projected down AB with an initial velocity of 24 f/s at A. AB is

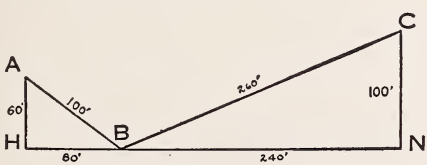


FIG. 239.

rough ($\mu=1/10$) ; no shock occurs at B, and the weight continues up the rough grade BC ($\mu=1/20$). Find the K. E. of the weight when it is half way up BC and still rising. (Fig. 239.)

643. A cylinder of weight W rests on a rough horizontal table with axis parallel to and s feet from the edge. A cord is wound around it, passes over a smooth pulley at the edge of the table and is attached to a weight H. Find how long it will take the center of the cylinder to reach the edge of the table.

644. Find the work done by a locomotive that changes the velocity of a train weighing 200 tons from 25 m/h to 50 m/h in 2 minutes, the frictional resistance being 10 pounds per ton.

645. A wheel is making 200 R. P. M., and after 10 seconds its speed has fallen to 150 R. P. M. If the angular retardation be constant, how many more revolutions will it make before coming to rest?

646. If it takes 600 useful H. P. to draw a train of 335 tons up a grade of 1 in 264 at a uniform speed of 40 m/h, estimate the resistance per ton other than that due to ascending against gravity, and find the uniform speed on the level when developing the above power.

647. A uniform circular plate, 1 foot in diameter and weighing 4 pounds, is hung in a horizontal plane by three fine parallel cords from the ceiling, and when set in small torsional vibrations about a vertical axis is found to have a period of 3 seconds. A body is laid diametrically across it and the period is found to be 5 seconds, the weight being 6 pounds. Find the moment of inertia of the body about the axis of oscillation.

648. The mass of a flywheel may be assumed to be concentrated in the rim. If the diameter is 7 feet and the weight 5/2 tons, estimate its K. E. when running at 250 R. P. M. If the shaft is 6 inches in diameter and the coefficient of friction of the shaft in the bearings is .09, find the number of revolutions the flywheel will make before coming to rest.

649. If a hammer whose weight is W, falling through a height h, strikes a pile of weight P, and drives it into the clay or sand, show that the energy available for penetration is $W/(W+P)\cdot Wh$, supposing no rebound of the hammer. If $W=\frac{1}{2}$ ton, $P=\frac{1}{8}$ ton, $h=10$ feet, and the pile is driven 6 inches into the ground, find the mean resistance of the ground.

650. A projectile has a kinetic energy of 1,670,000 ft.-lbs. at a velocity of 3000 feet per second. Later its velocity is only 2000 feet per second. How much K. E. has it lost?

651. An 80-ton gun discharges an 800-pound shot with a velocity of 2200 f/s. If the recoil is resisted by a constant pressure of 15 tons, how far will the gun recoil?

652. Calculate the kinetic energy of rotation of a projectile if its weight is 12 pounds, its radius of gyration is 0.75 inches, and its speed of rotation is 500 R. P. S.

ANSWERS

2. 7, 21° 47′.

7. 7.25 mi., N. 88° 05′ E.

9. (a) 14.53, 80° 06′, 58° 55′, 147° 04′; (b) 9.66, 37° 05′, 108° 05′, 121° 09′.

10. (a) 47.42, 335° 03′; (b) 16.76, 72° 39′; (c) 39.56, 163° 51′; (d) 30.15, 264° 17′.

11. $\sqrt{3}$ perpendicular to second displacement.

12. $\sqrt{13}$.

13. $-3/\sqrt{5}$.

14. 46.35.

15. 6.32, 161° 34′.

16. 13 lbs., 22° 37′.

17. $\sqrt{2}$ lbs., S. W.

18. P lbs. perpendicular to given force P.

19. 20 lbs., 30° 31′ with 13 lb. force.

20. 2 lbs.

21. 18.03 lbs., 17° 26′.

22. 60°.

23. 6.46 lbs.

24. 150° 19′.

25. 3 lbs., 4 lbs., 5 lbs.

26. 11.5 lbs., 23.1 lbs.

27. 3 lbs.

28. 2.5 tons.

29. 64.76 lbs.

30. 5 lbs., 13 lbs.

32. $T = Wl/2\sqrt{l^2 - a^2}$.

33. $P = 4/3\ W$, $T = 5/3\ W$.

34. $C = 125$ lbs., $T = 75$ lbs.

35. $C = 3.86$ tons, $T = 2.83$ tons.

36. $F = \frac{1}{2}P \tan \theta$.
37. 267.5 lbs.
38. 74.4°.
39. 2.89 ft., 0.85 ft.
40. 17.32 lbs.
41. $W\sqrt{2Rh - h^2}/(R-h)$.
42. 112 lbs.
44. 62.6 lbs., 42° 11′.
45. 14.15 lbs., 77.95°.
46. 500 lbs., 300 lbs.
47. (a) 9 lbs., (b) 89.55 lbs., (c) 1/10.
48. 12.68 lbs., 61.96 lbs.
49. 53° 08′, 160 lbs.
50. (a) $W = 25.98$ lbs., $N = 15$ lbs.; (b) $P = 6.65$ lbs., $N =$ 5.94 lbs.
51. 20 tons.
52. 19.62 lbs.
53. $\tan^{-1} \mu$.
54. 21.05 lbs.
59. 6.52 lbs.
60. 30°, 0.577.
61. $\frac{1}{3}$, $10\sqrt{10}$ lbs.
62. $\tan \alpha = \dfrac{\mu_1 W_1 + \mu_2 W_2}{W_1 + W_2}$, $T = \dfrac{W_1 W_2}{W_1 + W_2}(\mu_1 - \mu_2) \cos \theta$.
63. $F\dfrac{\sin(\phi + \alpha)}{\cos \alpha + \mu \sin \alpha}$.
64. $5\sqrt{3}$ lbs., 5 lbs.
65. (1) $P/\sqrt{2}$; (2) 12/13 P.
66. 50 lbs.
67. 36.60 lbs., 44.83 lbs.
72. (b) $1 : 1 : \sqrt{3}$.
73. $\sqrt{3} : 1 : 2$.

74. 90° between $5W$ and $12W$, 112° 37′ between $12W$ and $13W$, 157° 23′ between $13W$ and $5W$.

75. 226° 34′ and 151° or 133° 26′ and 209°.

77. 40 lbs.

78. 104° 29′.

79. $2\sqrt{3}$, $\sqrt{3}$ or 6, $3\sqrt{3}$.

81. 34 lbs., 20 lbs.

82. 525 lbs., 450 lbs.

83. 3 lbs., 13 lbs.

84. (a) $P=R=5.77$ lbs.; (b) $Q=10$ lbs., $R=8.66$ lbs.; (c) $Q=3.92$ lbs., $R=5.28$ lbs.

85. (a) $R=30$ lbs., $Q=26$ lbs.; (b) $P=2570$ lbs., $R=3760$ lbs.; (c) $P=Q=1.45$ lbs.

86. $P/2$, $P/2$.

87. $P:Q=\sqrt{3}:1$.

88. $5\sqrt{2}$ lbs. in leg of rt. triangle; 10 lbs. in cord inclined at 30°; $5(\sqrt{3}-1)$ lbs. in hypotenuse.

91. 2930 lbs., 2070 lbs.

92. 282 lbs.

95. 10 lbs., 6 lbs.

96. 5759 lbs.

97. $10\sqrt{10}$ lbs., $\tan^{-1} 3$.

98. (a) 16 lbs. 10″ from 10 lb. force; (b) 4 lbs. 27″ from 10 lb. force.

99. 2 lbs., $\bar{x}=-1.5''$, $\bar{y}=2.5''$.

100. $10\sqrt{2}$ lbs.

101. 31.4 lbs.

102. A 29.6 lbs., B 49.6 lbs., C 20.8 lbs.

103. 204.0 lbs., 186.2 lbs., 146.2 lbs.

104. 18 lb.-ft.

105. 4 lbs., 2 lbs., $20\frac{4}{7}$ lbs., $4\frac{1}{5}$ lbs.

106. $4\frac{3}{8}$ ft., 6 ft., $22\frac{1}{2}$ ft., $40\frac{4}{5}$ ft.

107. 17.2 lbs.

108. 81 lb.-ft., 9 ft.
109. 16 lbs., $17\frac{3}{4}$ ft. from O.
110. 15 ft.
111. 4 ft.
112. $4\frac{4}{7}$ ft.
113. 4 lbs., 8 lbs., 12 lbs.
114. $13\frac{1}{12}$ lbs.
115. 20 lbs. 20.8 ft. from O.
116. 27 in.
117. $2\sqrt{2}\,Pa$.
118. $130\sqrt{3}$ lbs.
119. 36 lbs., 27 lbs.
120. 130 lbs., $10\sqrt{105}$ lbs.
121. $T=21.7$ lbs., $P=33.4$ lbs., $\theta=19.7°$ with vertical.
122. $BC=846$ lbs. (T), Reaction at $A=630$ lbs., $\theta=47°\ 50'$ with horizontal.
123. $T=10$ lbs., $R=22.4$ lbs., $\theta=63°\ 26'$ with horizontal.
124. 1167 lbs. (C).
125. 3000 lbs. (C).
126. 1.20 tons (T).
127. 410 lbs. (C).
128. Horiz. 9000 lbs. (C), diag. 707 lbs. (T).
129. Upper 6184 lbs. (T), mid. 2500 lbs. (T).
130. $Aa=Ba=500$ lbs. (C), $Ca=400$ lbs. (T).
131. $Aa=2125$ lbs. (C), $ab=0$.
132. $Bb=1200$ lbs. (C), $Cb=1500$ lbs. (T).
133. $Aa=2600$ lbs. (C), $ab=560$ lbs. (T), $Da=2400$ lbs. (T).
134. $Aa=20$ tons (C), $Bb=16\sqrt{2}$ tons (C), $ba=CD=7$ tons (T), $Da=Cb=16$ ton (T).
135. $ab=CD=4200$ lbs. (T), $Da=3375$ lbs. (T), $Bb=5625$ lbs. (C).

136. $BE = 3900$ lbs. (T), $CE = 6000$ lbs. (C), $A_H = D_H = 1760$ lbs., $D_V = 3300$ lbs.

137. $AC = 17$ tons (C), $E_H = A_H = 24$ tons, $E_V = 10$ tons, reaction at B on $BD = 9.43$ tons, inclination with BD 212°.

138. $AC = 8081$ lbs. (T), $E_V = 2000$ lbs., $E_H = 10{,}000$ lbs., $FD = 14{,}142$ lbs. (C), $B_V = 13{,}714$ lbs., $B_H = 5714$ lbs.

139. $B_H = 5000$ lbs., $B_V = 8000$ lbs., $C_H = 5000$ lbs., $C_V = 1500$ lbs., $D_H = 5000$ lbs., $D_V = 2000$ lbs.

140. 200 lbs., $200\sqrt{3}$ lbs., 30°.

141. 200 lbs., 250 lbs., $\tan^{-1} \frac{3}{4}$.

142. $ab = 300\sqrt{2}$ lbs. (C).

143. $R_1 = R_2 = 16$ tons, $Da = Dc = 50.6$ tons (C), $Aa = Bb = Cc = 48$ tons (T), $Db = 48$ tons (C), $ab = bc = 16$ tons (C).

145. $Bb = 40.2$ tons (C), $Ab = 32.8$ tons (T), $ab = 32.8$ tons (C), $Aa = 56.8$ tons (T), $Ca = 28.4$ tons (C), $W = 49.2$ tons, $BC = 61.2$ tons.

146. $Aa = 1058$ lbs. (C), $Da = 908$ lbs. (T), $ab = CD = 360$ lbs. (T), $Bb = 1135$ lbs. (C).

147. $be = 25$ tons (C), $ef = 0$, $cd = 30$ tons (C).

148. $bc = 0$, $de = 1270$ lbs. (T), $Be = 635$ lbs. (T).

149. $bd = 4\frac{1}{4}$ tons (C), $Dd = 12$ tons (T).

150. $Bb = 9\frac{3}{4}$ tons (C), $bd = 4\frac{1}{4}$ tons (T), $dc = 8$ tons (C).

151. $Aa = 1700$ lbs. (C), $Cb = 1500$ lbs. (T).

152. Reaction at $C = 1250$ lbs., reaction at B and $D = 5150$ lbs.

153. Reaction at $C = 2063$ lbs., at B and $D = 5970$ lbs.

155. h/3.

156. $2a(\sin A)/3A$.

157. 3h/4 from vertex.

158. $3a/8$ from base.

159. $\bar{y} = 5a/6$.

160. $\bar{x} = a(\sin A)/A$.

161. $2h/3$ from vertex.

162. $2h/3$ from vertex.
163. $\bar{x}=\bar{y}=4a/3$.
164. $(18/5, 18/5)$.
165. $a/4$ from base.
166. $(2a/5, 2a/5)$.
167. $(4a/5, 0)$.
168. $(5a/6, 0)$.
169. Bisects line joining centers of bases of zones.
170. $2(n+2)a/(n+4)$.
171. $\left(\frac{\pi}{2}, \frac{\int_0^\pi \sqrt{1+\cos^2 x}\sin x\,dx}{\int_0^\pi \sqrt{1+\cos^2 x}\,dx}\right)$, $(\pi/2, \pi/8)$, $(\pi/2, 0)$, $(\pi/2, 0)$.
172. $(\pi a/4, a/2, \pi a/8)$.
173. $3a/4$.
174. $50a/63$.
175. $(6\frac{6}{13}, 6\frac{8}{13})$.
176. 15/194 in. from center.
177. Center of hole is 16″ from center of disk.
178. 3080 miles.
179. $h(2-\sqrt{2})$.
180. $\frac{h}{3}\cdot\frac{2b+a}{a+b}$.
181. 1.57 calibers from base.
182. 5.12″ from base.
183. 5.54″ from base.
184. $\frac{1}{6}a$.
186. $S=\pi a^2\sqrt{3}$, $V=\pi a^3/4$.
187. $S=4\pi^2 ab$, $V=2\pi^2 a^2 b$.
188. $2a/\pi$, $4a/3\pi$.
189. $76.8\,\pi$.
190. $4\pi ab^2/3$.

191. $\pi^2 a^3/4$.
192. 7/3 by integration.
193. 118.8.
194. 6.802 π.
195. 20.4.
196. 3523.3 cu. ft.
197. 15.24 sq. in., 2.96 in. from base.
198. 14,180.
199. 49,405 cu. ft.
200. 450 sq. ft., 14,320 cu. ft., 17.73 ft.
201. $bh^3/12 = Mh^2/6$.
202. $3/10\, Mr^2$.
203. $\frac{2}{5} Ma^2$.
204. $\frac{21}{32} \pi a^4$, $\frac{35}{16} \pi a^4$.
205. $21/512\, \pi a^4$.
206. $Mb^2/4$.
207. $4/15\, ab^3$, $4a^3b/7$.
208. $\pi ab^4/6$.
209. 9369 in.^{4}, 94.64 in.^{2}.
210. 1120/3 in.^{4}, 28/3 in.^{2}.
211. $\dfrac{bh^3 - b_1 h_1{}^3}{12}, \dfrac{bh^3 - b_1 h_1{}^3}{12(bh - b_1 h_1)}$.
212. $\dfrac{\pi(R^4 - r^4)}{4}, \dfrac{R^2 + r^2}{4}$.
213. $\dfrac{\pi h}{2}(R^4 - r^4), \dfrac{R^2 + r^2}{2}$.
214. $\frac{1}{6}\pi r^4 h$, $r^2/3$.
215. $h^3/3$, $h^2/3$.
216. $\pi r^3/2$, $r^2/2$.
217. $5/4\, \pi r^4$, $5/4\, r^2$.
219. 1948 in.^{4}, 54.11 in.^{2}.
220. $A = 524.25$ sq. ft., 58,819 ft.^{4}, 112.19 ft.^{2}.
221. $\frac{1}{6}a^5$, $\frac{1}{6}a^2$.

222. $1/12\, abh(a^2+b^2)$, $1/12\ (a^2+b^2)$.

223. $2\pi^2r^2R(R^2+\frac{3}{4}r^2)$, $(R^2+\frac{3}{4}r^2)$; $2\pi^2r^2R(R^2/2+\frac{5}{8}r^2)$, $(R^2/2+\frac{5}{8}r^2)$.

224. $\dfrac{\pi a^2h}{20}(a^2+4h^2)$, $\dfrac{3}{20}(a^2+4h^2)$; $\dfrac{\pi a^2h}{20}(a^2+h^2/4)$, $\dfrac{3}{80}(4a^2+h^2)$.

225. $\pi a^4/8$.

226. $25\pi a^4/192$.

227. $4/21\ \pi h^4b$, $5h^2/14$.

228. $\frac{101}{60}\pi a^5$, $\frac{101}{60}a^2$.

229. $\frac{4}{15}\pi abc(b^2+c^2)$, $\frac{1}{5}(b^2+c^2)$.

230. $\pi a^6/6$, $a^2/3$.

231. 180 in.^{4}.

232. $432\ \pi$ in.^{4}.

233. 0.675 lb.-ft.^{2}

234. 11.72 lb.-ft.^{2}.

235. 17.4 lb/in^2.

236. 374.5 lbs.

237. 14,976 lbs.

238. Midway.

239. $wr^2h(2+\pi/2)$ lbs.

240. 468,750 lbs.

241. 5000 lbs.

242. 10,056 tons.

243. $5\pi wa^3/2$ lbs.

244. 61.1 ft.

245. 7.74′.

246. 4670 lbs.

247. 5280 lbs.

248. 2880 ft.

249. 2275 lbs.

250. $2h/3$.

251. (a) $h/2$, (b) $3h/4$.
252. $3r/8$ from vertical and $3\pi r/16$ deep.
253. $3\pi b/16$.
254. (a) $4h/7$, (b) $5h/7$.
255. $h(a+3b)/2(a+2b)$.
256. Each top hinge 937.5 lbs., each bottom hinge 4687.5 lbs.
257. 2.03 diameters.
258. 1031 lbs. per linear ft., at 2.82 ft. above the sill.
259. 12 ft.
260. 844 lbs. at base.
261. 2.95 ft.
262. 2.4 ft.
263. 18.6 ft.
264. 9.78 ft.
265. $28.8w$ lbs., 2.61 ft. from A and .787 ft. from AB.
266. $\pi r^3 w$ lbs., at depth of $5r/4$ ft.
267. 2437.5 lbs., at depth of 1.95 ft.
268. $5\pi r^4 w/4$ lb.-ft.
269. 32,832 lbs., at depth of 11.21 ft.
270. 900 lbs.
271. 2667 lbs. per sq. in.
272. 2 : 1.
273. 19.76 ft.
274. $7ab/6\sqrt{a^2+b^2}$.
275. 5600 lbs.
276. $72w$ lbs., 4 ft. below the surface.
277. 23.6 ft.
278. (a) 8·7 ft., (b) 8·4 ft.
279. 955.7 lbs.
280. (a) $6w$ lbs., (b) $3w$ lbs.
281. 1.69 ft.
282. 122,061.5 lbs.
283. 5.63 ft.

284. 29,600 lbs., 35.4″ below surface, and 35.1″ from left end.
285. r.
286. 10,186 lbs. per sq. in.
287. 4096 lbs. per sq. in.
288. 6250 lbs. per sq. in.
289. $\frac{1}{4}''$ by $\frac{1}{4}''$ in cross section.
290. 0.17″.
291. 79 ft. $6\frac{1}{2}$ in.
292. 17,670 lbs.
293. 17,630 ft.
294. 14,710,000 lbs. per sq. in.
295. 4.9, 14.6, 9.6.
296. 49.6, 58,900 lbs.
297. 1042 lbs. per sq. in., 48.
298. 6875 lbs. per sq. in., 5.8.
299. Fail by rupture of plate.
300. 60%.
301. $\pi q d^2 = 2(p-d)tf$ where f = tensile stress.
302. 3,394 lbs. per sq. in.
303. 2.16″.
304. 2782.6 lbs. per sq. in., 9050.5 lbs. per sq. in., 72%.
305. 0.39″.
306. 200 lbs., $83\frac{1}{3}$ lbs.
307. 581 lbs., -419 lbs., -959 lbs.
309. 160 lbs. 4′ from left end, -340 lbs. 4′ from right end.
310. 4500 lbs., 500 lbs., -6500 lbs.
311. 3.15 tons.
312. 120 lbs.
313. At $x=6$, 165 lbs., 2490 lb.-ft.; at $x=10$, 165 lbs., 3150 lb.-ft.; at $x=15$, -285 lbs., 2175 lb.-ft.
314. 7600 lb.-ft., 7600 lb.-ft., 3120 lb.-ft.
315. At $x=10$, 860 lbs., 12,600 lb.-ft.; at $x=19$, -860 lbs., 12,600 lb.-ft.; at $x=26$, -1420 lbs., 5340 lb.-ft.

319. $14\frac{2}{3}$ ton-ft.
320. 1500 lb.-ft.
321. Symmetrical; one wheel over a support.
322. $26\frac{2}{3}$ ton-ft.
323. 5.6 times as strong.
324. Breadth $d/\sqrt{3}$, height $2d/\sqrt{3}$, $64:9\sqrt{3}\pi$.
325. 520 lbs.
327. $(r_1^2+r_2^2):r_1\sqrt{r_1^2-r_2^2}$, $(r_1^4-r_2^4):r_1^4$.
328. 8.
330. 8.64″.
331. 574.4 lbs.
333. 3″ × 10″ or 2″ × 12″.
335. 8.3, 5.6.
336. 12 in. deep. 35 lbs. per ft. (steady load).
337. 9 in. deep, 30 lbs. per ft. stationary. 12 in. deep, 40 lbs. per ft. rolling.
341. −380 lbs.
342. $\dfrac{r_1^2+r_2^2}{r_1\sqrt{r_1^2-r_2^2}}$.
343. hollow/solid = 3.64.
344. 782.15 lb.-ins.
345. 6484 lbs. per sq. in.
346. 19.98 in.
347. 12,011,000 lbs. per sq. in.
348. 10.6″ × 14.1″.
349. 30×10^6 lbs. per sq. in.
350. 298.2 tons.
351. 1.47″.
352. 2100 ft.
353. 2.22″, 66%.
354. $2\frac{1}{7}$ tons per sq. in.
355. $3\frac{3}{4}$ tons per sq. in.

356. 792 lbs.
357. 24 tons per sq. in.
358. 239 ft. 2 in.
359. Depth 17.32″, breadth 10″.
360. 11,230 lbs. per sq. in.
361. 117 lbs.
362. -320 lbs.
363. 1200 lbs.
364. 2080 lbs., 12 ft. from left end.
365. ± 144 lbs. at the center.
367. 2400 lb.-ft., 3600 lb.-ft.
368. 2560 lb.-ft., 1120 lb.-ft.
369. -7200 lb.-ft., -800 lb.-ft.
370. 1.82″.
371. 13.36 tons per sq. in.
372. 5260 lb.-ft.
373. $-466\frac{2}{3}$ lbs., 10.81 ft.
374. 14.70″.
375. 55 ft. 10 in.
376. 22 m/h.
377. $8\frac{1}{2}$ f/s, $5\frac{8}{11}$ m/h.
378. 35 m/h.
379. 1666.6 cm/min.
380. 8.046 kilo/hr.
381. $3an/88t$ m/h.
382. 85 f/s, 40 f/s, 5 f/s.
383. $0, -5\pi/2, -5\pi, -5\pi/2, 0, 5\pi/2, 5\pi, 5\pi/2, 0$.
384. $5/108$ m/m^2.
385. 11 f/s^2.
386. 2.93 f/s^2.
387. 0.22 f/s^2.
388. $a+2bt+3ct^2$, $2b+6ct$.

389. $-ka \sin (b+kt)$, $-k^2a \cos (b+kt)$.

391. -2 f/s, -0.5 f/s, 0.4 f/s², 0.025 f/s².

392. $2\sqrt{(5-s)(s-3)}$, $4(4-s)$.

393. 40 secs.

394. 18 f/s².

395. 50 sec., 25 metres.

396. 5 sec., $12\frac{1}{2}$ ft.

397. 16 f/s², 30 f/s.

398. $\frac{1}{3}$ sec., $\dfrac{\sqrt{2}-1}{3}$ sec., $\dfrac{\sqrt{3}-\sqrt{2}}{3}$ sec.

399. 50 m/h.

400. 24 f/s.

401. 9 f/s.

402. 6.25 f/s.

403. *A*, 0; *B*, 9/2 of its former velocity.

404. $\frac{1}{2}$.

405. $9\frac{1}{6}$ f/s, $18\frac{1}{3}$ f/s.

406. $\sqrt[4]{\frac{1}{2}}$.

408. $v = -\frac{1}{2}cs + he^{-\frac{1}{2}ct}(a \cos ht - b \sin ht)$,
$a = -cv - (h^2 + c^2/4)s$.

409. 25 ft., $\frac{1}{4}$ sec., $2\frac{1}{4}$ sec.

410. 545 cm/sec, 4/9 sec.

411. 10.2 sec.

412. 96 f/s, 0.

413. 20×10^3 dynes.

414. 49.05 kg.

415. $2^m\ 56^s$.

416. 110 lbs.

417. 25 : 4.

418. $6\frac{1}{4}$: 1, 5/56 f/s².

419. $133\frac{1}{3}$ f/s.

420. 5.0625×10^9 dynes.

421. 154 lbs., 70 lbs.
422. $2057\frac{1}{7}$ ft.
423. 200 ft., 5 sec.
424. $30°$.
425. $g(\sin\alpha-\mu\cos\alpha)$, $\sqrt{2gl(\sin\alpha-\mu\cos\alpha)}$.
427. $g/5$, $7\frac{1}{5}$ lbs.
428. $m/2$.
429. 2 lbs.
430. 5 oz.
431. $57\frac{1}{7}$ ft., $10\frac{2}{7}$ lbs.
432. 24 lb. 10 oz.
433. $\mu=0.3$.
434. $s=a\sin k(t-t_0)$.
435. Period $=2\pi/k$.
436. ka.
438. $\pi/2$, 3, π, 12.
439. $a\cos kt_0$, $-a\sin kt_0$.
440. 2 in., 0.453 sec.
441. $s=5\sin\left(\frac{2\pi t}{3}-\frac{\pi}{3}\right)$.
442. 42 min. 20 secs., 4.93 mi/sec.
444. $s=1/k\log(1+kv_0t)$.
445. No.
446. 2 f/s.
447. $(m_1-m_2)^2g/(m_1+m_2)$.
448. 29 ft. 9 in. nearly.
449. $5\frac{20}{27}$ tons.
450. 1 mi. 1408 yds.
451. 784 ft.
452. $3\frac{1}{3}$ sec.
453. Distance 6, force $\frac{1}{2}\pi^2(7-2s)$.
454. $\frac{1+e^2}{1-e^2}h$, $\frac{1+e}{1-e}\sqrt{2h/g}$.

455. $(5/9)^4 v$.

456. 19 : 13.

457. $2\frac{1}{2}$ lbs., $3\frac{1}{3}$ lbs., $g/6$.

458. $(m - m' \sin \alpha) g/(m + m')$.

459. (a) 500 ft.-lbs., (b) no.

460. 65.82 ft.-lbs.

461. 24,062 ft.-lbs.

462. 352,000 ft.-lbs.

463. 112,500 ft.-lbs.

464. 8,906,250 ft.-lbs.

465. 1875 ft.-lbs.

466. 323.5 ft.-lbs. by P, 250 ft.-lbs. against R, 500 ft.-lbs. against gravity.

467. 47,920 ft.-lbs.

468. 68,760 ft.-lbs.

469. 52 ft.-lbs., 130 ft.-lbs.

470. (1) 48.8 ft.-lbs., (2) 195 ft. lbs.

471. $W(\mu b + h)$ ft.

472. 5808 ft.-lbs.

473. 300 ft.-lbs.

474. 338 ft.-lbs.

475. 1.49×10^5 ft.-lbs.

476. $8W$ ft.-lbs.

477. 42,500 ft.-lbs.

478. 105 ft.-lbs.

479. 500,000 ft.-lbs.

480. 93.3 ft.-lbs.

481. 14,242 ft.-lbs.

482. 4.22 ft.-lbs.

483. 26,400 lbs.

484. $426\frac{2}{3}$ H. P.

485. 508.8 H. P.

486. 3.584 H. P.

487. 160 H. P.
488. 409 H. P.
489. 0.20 f/s^2.
490. (a) 4,400,000 ft.-lbs., (b) 30 H. P.
491. (a) 5.05×10^{10} ft.-lbs., (b) 7.65×10^6 H. P
492. $2^m\ 17.5^s$, 1512.5 ft.
493. 4.97 ft.
494. 300.6 ft.
495. 375.7 ft.
496. 174.5 ft.
497. 448.8 ft.
498. 742,500 ft.-lbs.
499. $w\pi a^2h^2/12$.
500. 11,778 ft.-lbs.
501. 1000 lbs.
502. 250,000 ft.-lbs.
503. 24,375 ft.-lbs.
504. 320 H. P.
505. 11.95 H. P.
506. 380 H. P.
507. 12 H. P.
508. 742.4 H. P.
509. 60.7 H. P.
510. 63.1 H. P.
511. 264 H. P.
512. $\mu = .163$.
513. $\frac{1}{2}$.
514. 20.4.
515. $\dfrac{.293}{a}$ ft.-lbs.
517. $w \cdot a$.
518. $\frac{1}{3}\sqrt{209}$ f/s, 32/9 f/s^2.
519. $v_x = 16/\sqrt{37}$ f/s, $v_y = 60\sqrt{37}$ f/s.

520. $\tan \tau = 24$.
521. $v_y = -6\sqrt{5}$ f/s, $a_y = -162$ f/s^2, $t = \frac{1}{2}$ sec.
522. $a_x = 4x$, $a_y = 4y$.
524. $v = 4$, $a_t = 0$, $a_n = 8\sqrt{2}$.
525. $a_t = -\sqrt{2}$ f/s^2, $a_n = \sqrt{2}$ f/s^2.
526. $6\pi/5\ r/s^2$, $t = 5$ sec.
527. $600\pi\ r/m^2$, $\theta = 75/2\pi^2$ rads.
528. $-100\pi\ r/m^2$, 200π rads.
529. (a) 402, (b) 25.1 sec.
530. $-ak^2 \cos kt$, $-ak^2 \sin kt$, 0, ak^2.
532. $v_x = 4$ f/s, $v_y = 4\sqrt{3}$ f/s.
533. 7.2 f/s^2.
534. $58\frac{2}{3}$ f/s, $a = 491.7$ f/s^2.
535. 1536.5 f/s, 0.111 f/s^2.
536. $2\pi/5\ r/s^2$, 15π f/s, 47.5π f/s^2.
537. (a) $-6\pi/5\ r/s^2$, (b) $40\frac{5}{6}$, (c) $588\pi^2$ f/s^2.
539. 2771.2 ft.
540. $x = 886.8$ ft., $y = 256$ ft.
541. 45°.
542. $v_0 = 12\sqrt{2}$ f/s, 340 ft.
543. 150 ft.
544. 288 ft., $3\sqrt{2}$ sec.
545. 160 f/s, $\tan^{-1}(-4/3)$.
546. $333\frac{1}{3}$ f/s.
547. 6250 ft.
548. 300 ft. nearly, $t = 1$ sec.
549. $37{,}812\frac{1}{2}$ ft.
550. 5135 f/s, 51° 38′.
551. $8\sqrt{3}$ ft.
552. 52.7 lbs.
553. 1 3/11 R. P. S.
554. 0.4949 ft.

555. 1 hr. 25 min.

556. $\cos^{-1} 0.6079 = 52° \ 34'$.

557. 1.17 ft., 205.6 lbs.

558. $2\pi\sqrt{2/g}$ sec., $\frac{1}{2}l$ lbs.

559. 8° 36′.

560. 2°, 1.95 in.

563. 3.39 in.

564. 23.39 tons.

565. 12 rad/sec.

566. $\pi r^4 \omega^2 w/4g$.

567. $\omega^2 = \dfrac{4g}{\pi r^4}\left(\dfrac{2\pi r^3}{3} - V\right)$.

568. 39.15 in.

569. 32.13 f/s.

570. 59.33.

571. $117\frac{1}{3}$ ft.

572. $g_1 : g_2 = 125 : 126$.

573. $12\sqrt{2}$ f/s.

574. $4\sqrt{2}$ f/s, $8\sqrt{10}$ f/s.

575. $22\frac{1}{2}$ lbs.

576. $16\sqrt{3}$ f/s.

577. $\sqrt{l(\cos\theta - \cos\theta_0)}$.

578. 53° 8′.

581. $1.4616r$ from the lowest point of the circle.

582. $x/a = y/b$, $\alpha_x = k^2 x$, $\alpha_y = k^2 y$.

584. $324/x^3$, $-(16/y)^3$.

585. $v_x = ky/p$ (p is the parameter of the parabola), $\alpha_x = k^2/p$.

587. $7\frac{1}{2}$ lbs.

588. (a) 1.5 sec., (b) 25.1 from building, (c) 59.7 f/s at 16.5° to the vertical.

589. 137.5 ft. from the vertical starting point, $6\frac{1}{4}$ sec., 201 f/s at $6\frac{1}{3}°$ to vertical.

590. $a_t = -\frac{1}{60}\sqrt{10}$, $a_n = \frac{1}{60}\sqrt{290}$.
592. (a) 0.022 r/s, (b) 15.7 f/s, (c) 7.85 f/s.
593. 196 ft.
594. Arc tan $\frac{2}{3}$ or arc tan 14/3.
595. dr/dt, $r\,d\theta/dt$.
596. $46\frac{5}{104}$ ft.-lbs.
597. 6.52 in.
599. 6.16 lbs. per ton.
600. 302.5 ft.
601. 41 m/h (nearly).
602. 1053 H. P.
603. 19.2 m/h.
604. 90%.
605. 7.23 in.
606. 19,950 ft.-lb., 241.8 H. P.
607. 7.37 secs.
608. 5/3 lbs.
609. $9\frac{1}{7}$ f/s^2, $18\frac{2}{7}$ r/s^2, $14\frac{2}{7}$ lbs.
610. $\beta = 0.335 r/s^2$, $T = 49.91$ lbs., $s = 0.698$ ft., $a = 0.056$ f/s^2.
611. 23 rev.
612. $\pi\sqrt{11a/6g}$.
613. 2.3338 ft.
614. 92.6 beats per min.
615. $h = r$.
616. 0.463 sec., 0.7 ft., 0.2 ft. below center of sphere.
617. 0.866 sec., 2.44 ft.
618. 125 (nearly).
619. 3300 lb.-ft.^{2}.
620. 48.7 ft.-lbs.
621. 44.3 revolutions.
622. $20{,}600k^2$ ft.-lbs., 10.5 revolutions.
623. 2×10^{29} ft.-lbs. (approx.).

624. 208 tons.

625. 3367.3 H. P.

626. 14.57 in.

627. External diam. 15.09 in., internal diam. 8.49 in., 8893.5 lbs.

628. $4\frac{1}{2}$ tons, $98\frac{3}{5}$%.

629. 96 f/s, 432 ft., 19.59 f/s.

630. 280 ft.

631. Sphere, disk, hoop.

634. (a) 11.7 f/s downward, (b) 5.9 r/s, (c) $\frac{224}{65}$ r/s^2, $\frac{224}{65}$ f/s^2, $\frac{448}{65}$ f/s^2, (d) 1.703 sec.

635. 1,391,500 ft.-lbs.

636. 26.89 ft.-tons.

637. 2611 f/s, 6596 ft.-lbs., 30.37 lbs.

638. 8 f/s.

639. 250 ft.-lbs.

640. 139.57 tons, 271.38 ton-ft.

641. $7\frac{7}{9}$ lbs.

642. 1000 ft.-lbs.

643. $\sqrt{\frac{(3W+8H)s}{2gH}}$ secs.

644. 18,497.03 ft.-tons.

645. 37.5 revolutions.

646. 8.3 lbs. per ton, 81 m/h.

647. 1/36 lb.-ft.^{2}.

648. 328.17 ft.-tons, 928 revolutions.

649. 8 tons.

650. 927,778 ft.-lbs.

651. 8.04 ft.

652. 7230 ft.-lbs.